Mathematical Control Strategy with Time-Dependent Contact Rate for Contagious Disease

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ABSTRACT
Contact between susceptible and infected individuals is one of the major reasons for the spread of contagious viral disease, for example, the severe acute respiratory syndrome, SARS, and is a major public health problem in the world. The present study aims to assess *via a* mathematical model, the role of contact rate in the control of the spread of contagious disease like SARS. In this article, we have induced an effective contact rate in the mathematical model as a periodic function of time due to the seasonal occurrence of SARS which was considered as a parameter earlier. The spread of the disease also depends on the time taken to initiate preventive measures by the authorities which have been described and explained by a new term, action time, in the present study. Numerical simulations have been performed with the help of fourth-order Runge-Kutta method to illustrate our results. With the help of simulation, the control of the spread of diseases has been explained with varying periodic effective contact rate and action time.

Keywords
SARS; SEIR model; effective contact rate function; simulation; action time; control of the disease.

1. INTRODUCTION
The contagious disease like SARS is transmitted by close contact from person-to-person worldwide [14]. The incubation period of SARS is 2 to 7 days, although in some cases it may be as long as 10 days [15]. The control of contagious disease like SARS is based on quarantine of infected individuals and isolation of individuals with clinical symptoms.

In classical epidemiology, a critical factor is known as 'mass action principle' which states that the course of an epidemic depends on the contact rate between susceptible and infected individuals. Also, the net rate at which infections are acquired is proportional to the numbers of encounters between susceptible and infected individuals [8]. And the constant of proportionality, denoted by β has been termed as the transmission coefficient [1]. As it has been reported that, the contagious disease like SARS spread with the close contact with the infected individuals [7] therefore to control the infection of the contagious disease like SARS we need to control the effective contact rate. In this article we have developed a SEIR model with the help of effective contact rate function which was considered as parameter in other previous models and also describe the spread control of the infection in the individuals with the help of action time. It is well-known that many disease exhibit seasonal (periodic) fluctuations, such as influenza, measles, whooping cough, etc. ([2], [6], [11]). Zhang et al. [16] have considered time varying periodic effective contact rate for rabies in china because of seasonal occurrence of disease.

Over the past few decades, In the mathematical biological literature, a large number of compartment mathematical models have been proposed to control the spread of infectious diseases such as SIS [9], SIR [2,10], SEIR [9,12], SEIRS [13], SVEIR [7] (where S, V, E, I and R denotes the population of susceptible, vaccinated, exposed, infected and recovered individuals respectively). The model constructed in the present article is an extension of the standard SEIR model, therefore an attempt has been made in the present paper to develop a SEIR model for the contagious disease like SARS with the induction of periodic effective contact rate function and also describe the spread control of the infection in the society with the help of action time. Action time or period has been introduced in this manuscript which may be defined as "A time taken by the health agencies to control the spread of infection from infected to susceptible individuals by various means such as by increasing the immunity of infected and susceptible, vaccination of both susceptible and infected, quarantine of infected, yoga etc." Including an effective contact rate as a periodic function, this was considered as a parameter earlier.

In this article, we simulate the data of SARS cases reported by Greater Toronto Area in 2003 [7].

The rest of paper is organized as follows: A SEIR mathematical model for control the spread of contagious viral disease like SARS is formulated in Section 2. Effective contact rate function is described in Section 3. Basic properties of solutions are given in Section 4 and numerical simulation and discussion are in Section 5. Finally, the conclusion is summarised in Section 6.

2. A SEIR MODEL FOR CONTAGIOUS DISEASE
In this section, we have established a SEIR model for the transmission of the contagious disease like SARS. For this, the total population is divided into four compartments: susceptible $S(t)$, exposed $E(t)$ infective $\mathrm{I}(t)$ and recovered $R(t)$.The parameters $\beta, \alpha, \delta, d \; and \; \mu$ are the effective contact rate, rate of development of clinical, recovery rate, disease induced mortality rate and natural mortality rate respectively. Our motive is to control the spread of SARS using mathematical model, for which we have assumed that susceptible individuals can be infected only through contact with infectious individuals. Therefore, instead of considering β as a parameter, we have formulated an effective contact rate which is a function of time t.The progression of infection through different compartments shown with help of block diagram, which is given below:

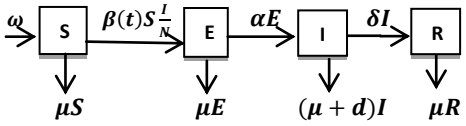

Fig. 1: Progression of infection from susceptible (S) through exposed (E) infected (I) and recovered (R) compartments for the model .

After incorporating these changes in standard SEIR model [12], the rate of change of the population in each compartment is given by the following system of differential equations

$$\frac{dS}{dt} = \omega - \mu S - \beta(t) S \frac{I}{N} \quad (1)$$

$$\frac{dE}{dt} = \beta(t) S \frac{I}{N} - \mu E - \alpha E \quad (2)$$

$$\frac{dI}{dt} = \alpha E - (\mu + d + \delta) I, \quad (3)$$

$$\frac{dR}{dt} = \delta I - \mu R. \quad (4)$$

where $S(0) = S_0, E(0) = E_0, I(0) = I_0, R(0) = R_0, t \geq 0$ and $\beta(t) > 0$ is an effective contact rate function. The total population size is $N(t) = S(t) + E(t) + I(t) + R(t)$.The effective contact rate function $\beta(t)$ is described in next section.

The explanation of above model parameters is listed in Table1.

Table 1: Parameters description and values used in simulation

Parameter	Description	Value	Source
ω	Recruitment rate	146 per day	Gumel et al. (2006)
μ	Natural mortality rate	3.65×10^{-5} per day	Gumel et al. (2006)
α	Rate of development of clinical symptoms	0.125 per day	Gumel et al. (2006)
N	Equilibrium Population	4000000	Gumel et al. (2006)
d	Disease-induced mortality rate	0.008 per day	Gumel et al. (2006)
δ	Recovery rate	0.04 per day	Gumel et al. (2006)
c	Action time	2 to 10 days	Assumption [15]
b	Spread controlling parameter	(c + 2) days	Estimation

3. EFFECTIVE CONTACT RATE $\beta(T)$

This section will be used to formulate effective contact rate function. The only way of transmission of Infectious disease is close contact between susceptible and infectious individuals and also the probability of getting a disease is not constant at any point of time. Since the occurrence of SARS is seasonal and prevalent during winters and its transmission is very fast, therefore contact rate will be considered as a periodic function of time. Following assumptions have been made to formulate the effective contact rate function:

1. It has been observed from the literature and data, that intensity of the infection of SARS goes up till a certain period of time.
2. The prime reason for the spread of disease is contact between infected and susceptible individuals and therefore, effective contact rate should increase with time in a periodic manner.
3. Also, it is considered that effective contact rate cannot be completely zero at any time t up to certain period of time.
4. A force of infection (P) will be considered with periodically ('seasonal') varying contact rate i.e.

$$P = \frac{\beta(t) I}{N}, \beta(t + T) = \beta(t) \quad (5)$$

With period T equal to one year.

Hence effective contact rate function $\beta(t)$ has been modeled as follows

$$\beta(t) = \frac{H(t) + c}{b}, 2 \leq c \leq 10, b = c + 2 \; and \; t \geq 0 \quad (6)$$

Where $H(t)$ is also a Periodic function of time with period T.

For the purpose of simulations effective contact rate function $\beta(t)$ has been modeled as follows:

$$\beta(t) = \frac{Sin^2 t + c}{b}, 2 \leq c \leq 10, b = c + 2 \; and \; t \geq 0 \quad (7)$$

where b represents the spread controlling parameter to minimize the infection of disease on the society and c represents an action time, which is "A time taken by the health agencies to control the spread of infection from infected to susceptible individuals by various means such as by increasing the immunity of infected and susceptible, vaccination of both susceptible and infected, quarantine of infected, yoga etc." We are taking the values of c greater and equal 2 days, which is the incubation period of SARS. It has been assumed that minimum action time to control the spread of the disease should not be less than incubation period, and therefore c can take minimum value as 2 days. The spread controlling parameter will always depend on the action time.

4. BASIC PROPERTIES OF THE MODEL

The model Equations $(1) - (4)$ monitors populations, it is assumed that all state variables and parameters of the model are nonnegative i.e. $(S, E, I, R) \in \mathbb{R}^4_+$ and $\omega, \mu, \alpha, d, \delta \geq 0$.

Theorem 1: The variables of the model (Equations $(1) - (4)$) are non-negative at all time.

Proof: Let $t_* = \sup\{t > 0 : S > 0, E > 0, I > 0, R > 0 \in (0, t)\}$. Thus, $t_* > 0$. It follows the equation (1) that

$$\frac{dS}{dt} = \Delta - \mu S - \beta(t) S \frac{I}{N}$$

which can be re-written as,

$$\frac{d}{dt}\left[S(t) \, e^{\left\{\mu t + \frac{1}{N}\int_0^t \beta(y) I(y) dy\right\}}\right] = \omega \, e^{\left\{\mu t + \frac{1}{N}\int_0^t \beta(y) I(y) dy\right\}}. \quad (8)$$

Hence, $S(t_*) \, e^{\left\{\mu t_* + \frac{1}{N}\int_0^t \beta(y) I(y) dy\right\}} - S(0)$

$= \int_0^{t_*} \omega \, e^{\left\{\mu \tau + \frac{1}{N}\int_0^t \beta(y) I(y) dy\right\} d\tau}$,

So that

$$S(t) = S(0)e^{-\left\{\mu t_* + \frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}} + \left[e^{-\left\{\mu t_* + \frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}\right]\int_0^{t_*} \omega\, e^{\left\{\mu\tau + \frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}d\tau > 0.$$

Similarly $E > 0,\ I > 0$ and $R > 0$.

Lemma 1: The closed set $D = \{(S, E, I, R) \in \mathbb{R}_+^4 : N \leq \frac{\omega}{\mu}\}$ is positively- invariant.

Proof:

The rate of change of the total population, obtained by adding Equations $(1) - (4)$, is given by

$$\frac{dN}{dt} = \omega - \mu N - dI. \quad (9)$$

Since $\frac{dN}{dt} \leq \omega - \mu N$, it follows that $\frac{dN}{dt} \leq 0$ if $N \geq \frac{\omega}{\mu}$. Thus, a standard comparison theorem for ODE can be used to show that $N(t) \leq \frac{\omega}{\mu} + (N(0) - \frac{\omega}{\mu})e^{-\mu t}$. In particular, $N(t) \leq \frac{\omega}{\mu}$ if $N(0) \leq \frac{\omega}{\mu}$. Thus, the region D is positively- invariant. Further, if $N(0) > \frac{\omega}{\mu}$, then either the solution enters in D finite time, or $N(t)$ approaches $\frac{\omega}{\mu}$ asymptotically. Hence region D attracts the all solutions in $\mathbb{R}_+^4$.

The system (Equations$(1) - (4)$) is continuous and its derivative implies that solutions exist and is unique. Since solutions approach lies in D they are bounded and hence exist for $t \geq 0$. Therefore, the model is epidemiologically and mathematically well posed.

5. NUMERICAL SIMULATIONS AND DISCUSSION

In this section, we have simulated the model numerically to understand the role of action time and other factors to minimise the effect of the disease. The numerical simulation of the model equations (1)-(4) has been done using Fourth order Runge- Kutta method [3, 5, and 4] in Matlab 2012b. The values of the parameters have been taken from table 1.

Since the total population N is 4, 00,000 [Table 1], therefore let $S_0 = 3999980$, $E_0 = 12$, $I_0 = 6$ and $R_0 = 2$ so that the sum of S, E, I and R will remain equal to the total population.

Since the aim of the present study is to assess the behavior of the susceptible, exposed and infected population with respect to the action time, therefore graphs for these populations have been drawn by taking different values of action time.

Figure 2, Figure 3, Figure 4 and Figure 5 show the population of susceptible, Exposed, infected and recovered individuals for c $= 3, 5, 7$ and 10 days respectively.

It can be seen from these figures that the peak values of exposed and infected population will be increasing when the values of action time are increased. Also, it can be observed from these graphs that the duration to achieve a decreased peak value will be increased with decreasing value of action time while the total duration for the eradication of the disease is almost same for all values of action time. It shows that if the time taken to take preventive measures is less, then the effect of the disease on the society will be reduced.

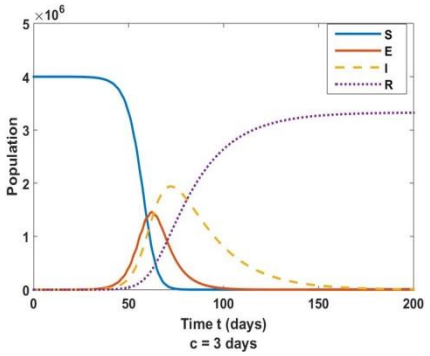

Fig. 2: SEIR population at action time $c = 3$ days

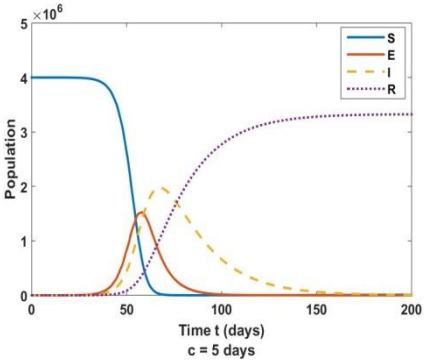

Fig. 3: SEIR Population at action time $c = 5$ days

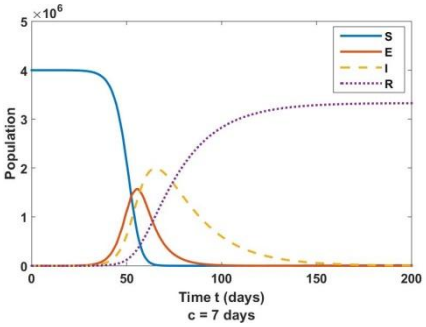

Fig. 4: SEIR Population at action time c $= 7$ days

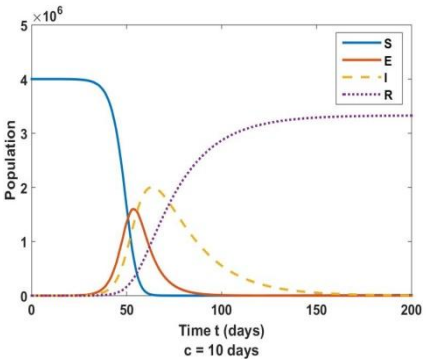

Fig. 5: SEIR Population at action time c $= 10$ days

Figure 6 shows the population of exposed individuals with four different values of action time. It is evident from the figure that when we take less time to initiate preventive measures, then the population of exposed individuals is less in comparison of greater time taken for initiation of preventive measures.

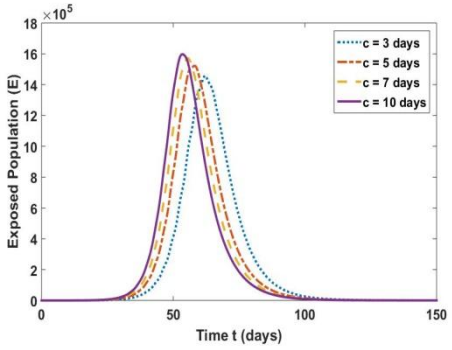

Fig. 6: Number of exposed individuals at various values of c.

The population of infected individuals at various values of action time $c = 3, 5, 7$ and 10 days is shown in figure 7. It can be observed from figure 7 that population of infected individuals increase with the increment in the value of action time. Also the total number of infected population is decreasing with respect to the decreased value of action time. It can be understood that if time to take preventive measures is increased, then more number of people will be infected in the society. Preventive measures may include various methods such as the decreased contact rate and quarantinization of the infected individuals.

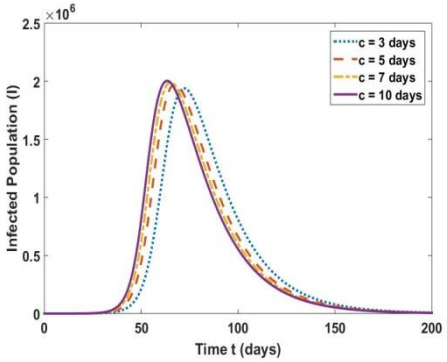

Fig. 7: Number of infected individuals at various values of c

Figure 8 shows the comparison of the infected population at a constant effective contact rate and effective contact rate as a function of time. For simulation, the value of parameter β has been considered as $7.2 \times 10^{-8}\ per\ day$ [7] and for function of time, the value of action time is taken as 10 days.

The figure shows that even the infected population has been calculated at the maximum value of action time of this study, i.e. 10 days, then also the infected population is much less than the infected population at a parametric value of effective contact rate. Also, it can be concluded that the duration to reach the infection at its peak value will be increased when effective contact rate is considered as a function of time. This time can be utilized in taking preventive measures, making people aware about the disease which will result in the prevention of the spread of disease.

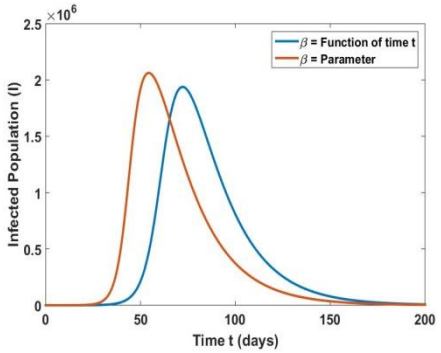

Fig. 8: Number of infected individuals for β as parameter and function

Figure 9 shows the transmission rate of SARS with time at different values of action time (in days). It can be seen that as time increases, the area under the curve for transmission rate with small action time is low in comparison of greater values of action time. Action time may be utilized to take preventive method to control the disease and also increasing the immunity of the susceptible to avoid them to become infected and of the infected to make a fast recovery.

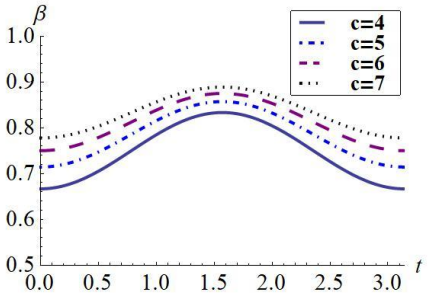

Fig 9: Effective contact rate $\beta(t)$ at different values of c.

6. CONCLUSION

Since experiments cannot be done with the lives of human beings for a disease, therefore, mathematical modeling plays a crucial role in understanding the dynamics of the disease and hence planning and evaluating interventions can be done in an effective manner. In this paper, we have presented a SEIR model for SARS, which is a contagious disease and cost many lives, by considering effective contract rate as a function of time to control the spread of the SARS infection with induction of new term, action time. It has been concluded from the simulation of the model that if the time taken to initiate preventive measures is increasing then more number of people will be caught by the infection and it will be difficult to control the spread of disease in the society which will cost heavy monetary expenditure along with precious human lives. Since real data is not available in the required format, therefore an attempt will be made to collect the real data and model will be validate accordingly. Also, the model will be studied for global behavior.

7. ACKNOWLEDGEMENT

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