Optimality and Duality for Nonlinear Program

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ABSTRACT
This proposed study deals with the optimality and duality results for non-linear convex programming problems, involving semi-differentiable functions with respect to a continuous arc.

Keywords
Fritz-John optimality criteria, real valued functions, convex functions, Weak and strong Duality.

1. INTRODUCTION
We obtain Fritz-John [5] type necessary optimality criteria for the optimal solution of the following non-linear program

(P) Minimize f(x)

Subject to $g_j(x) \le 0, \quad j = 1,2,...m,$

$x \in S$

Where $S \subseteq R^n$, $f: S \to R$ and $g_j : S \to R$,

$j = 1, 2, ..., m$ are real values functions, S is a locally connected set, such that for each $x^*, \; x \in S,$, there exists a vector valued function $H_{x^*,x^{(\lambda)}}$, satisfying

$$H_{x^*,x^{(\lambda)}}, \in S, \quad 0 < \lambda < a(x^*, x) \qquad (1)$$

$H_{x^*,x}$ is a continuous in the interval $]0, a(x^*,x)[$

and

$$H_{x^*,x^{(0)}} = x^*, H_{x^*,x^{(1)}} = x \qquad (2)$$

And the right differentials of f and g_j, $j = 1,2,...m$ at x* exist, with respect to the arc $H_{x^*,x^{(\lambda)}}$.

Let $X^\circ = \{x \in S | g_j(x) \le 0, \; j = 1,2,..m\}$

Theorem 1:

Let x* be an optimal solution of (P). If $(df)^+(x^*, H_{x^*,x^{(0+)}})$, and $(dg_1)^+(x^*, H_{x^*,x^{(0+)}})$

are convex functions of x, g_j, $j \in J$ is continuous at x* with S convex or $S = R^n$. Then there exist $r_\circ^* \in R, r^* \in R^m$, such that

$$r_\circ^*(df)^+(x^*, H_{x^*,x^{(0+)}}) + r^{*T}(dg)^+(x^*, H_{x^*,x^{(0+)}}) \ge 0 \quad \text{For all } x \in S \qquad (3)$$

$$r^{*T} g(x^*) = 0 \qquad (4)$$

$$(r_0^*, r^* \ge 0) \qquad (5)$$

where $I = I(x^*) = \{i | g_i(x^*) = 0\}$

and $\mathrm{J} = \mathrm{J}(\mathrm{x}^*) = \{ j | g_i(x^*) < 0\}$

Proof: Firstly, we shall show that the system

$$\begin{Bmatrix} (df)^+(x^*, H_{x^*,x^{(0+)}}) < 0 \\ (dg_I)^+(x^*, H_{x^*,x^{(0+)}}) < 0 \end{Bmatrix} \qquad (6)$$

has no solution, $x \in S$.

If possible, let $x \in S$ be a solution of the system (6). Since right differentials of f and $g_i, i \in I$ at x*, exist with respect to the arc (x^*, H_{x^*,x^λ}), therefore

$$f(H_{x^*,x^{(\lambda)}}) = f(x^*) + \lambda(df^+)(x^*, H_{x^*,x^{(0+)}}) + \lambda\alpha(\lambda) \qquad (7)$$

$$g_j(H_{x^*,x^{(\lambda)}}) = g_i(x^*) + \lambda(dg_i)^+(x^*, H_{x^*,x^{(0+)}}) + \lambda\alpha_i(\lambda) \qquad (8)$$

Where $\alpha : [0,1] \to R, \; \lim_{\lambda \to 0^+} \alpha(\lambda) = 0 \qquad (9)$

$\alpha_i : [0,1] \to R, \lim_{\lambda \to 0^+} \alpha_i(\lambda) = 0, \; \mathrm{i} \in \mathrm{I(x^*)} \qquad (10)$

Using (6),(9) and (10), we get, for small enough λ, say $0 < \lambda < \lambda_0$,

$$(df)^+(x^*, H_{x^*,x^{(0+)}}) + \alpha(\lambda) < 0$$

$$\text{and } (dg_i)^+(x^*, H_{x^*,x^{(0+)}}) + \alpha_i(\lambda) < 0, \quad i \in I(x^*)$$

Hence, it follows by using the relation (7) and (8) that for $0 < \lambda < \lambda_0$,

$$f(H_{x^*,x^{(\lambda)}}) - f(x^*) < 0 \quad (11)$$

and $$g_i(H_{x^*,x^{\lambda}}) - g_i(x^*) < 0, \quad i \in I(x^*) \quad (12)$$

Now, $g_j, j \in J$ is continuous at x* and $H_{x^*,x^{(\lambda)}}$ is also a continuous function of λ; therefore

$$\lim_{\lambda \to 0^+} g_j(H_{x^*,x^{\lambda}}) = g_j(x^*) < 0$$

which implies that there exist

$\lambda_j^*, 0 < \lambda_j^* < a(x^*, x), j \in J$, such that

$$g_j(H_{x^*,x^{(\lambda)}}) < 0, \text{for } 0 < \lambda < \lambda_j^* \quad (13)$$

Let $\lambda^* = \min(\lambda_0, \lambda_j^*, j \in J)$, then from (11) to (13), it follows that for $0 < \lambda < \lambda^*,\ H_{,x^*,x^{(\lambda)}} \in X^\circ$ and $f(H_{x^*,x^{(\lambda)}}) < f(x^*)$, which is a contradiction, as x* is an optimal solution of (P).

Hence, the system (6) has no solution $x \in S$.

Since $(df)^+(x^*, H_{x^*,x^{(0+)}})$, $(dg_I)^+(x^*, H_{x^*,x^{(0+)}})$ are convex functions of x, therefore, there exist $r_\circ^*, r_i^* \in R, i \in I$, such that

$$r_\circ^*(df)^+(x^*, H_{x^*,x^{(0+)}}) + r_1^{*T}(dg_1)^+(x^*, H_{x^*,x^{(0+)}}) \geq 0$$

$$\text{for all x} \in \text{S } (\text{r}_0^*, r^*) \geq 0$$

Defining r_j^* =0, we get the required result. Weir and Mond [10] provided a Fritz-John dual for the non-linear programming problems involving differentiable functions by using the Fritz-John optimality conditions instead of Kuhn-Tucker conditions and thus did not require a constraint qualification. Now, we associate the following Mond-Weir type Fritz-John dual to the problem (P):

(D) Maximize f(u)

Subject to $$r_0(df)^+(u, H_{u,x^{(0+)}}) + r^T(dg)^+(u, H_{u,x^{(0+)}}) \geq 0$$

$$\textit{for all } x \in X^\circ \quad (14)$$

$$\sum_{j=1}^m r_j g_j(u) \geq 0 \quad (15)$$

$$u \in S, (r_\circ, r) \geq 0, r_\circ \in R, r \in R^m \quad (16)$$

Theorem 2(Weak Duality) Let x be feasible for (P) and $(u, r_\circ, r)$ be feasible for (D). If *f* is locally p-connected and $\sum_{j=1}^m r_j g_j$ is strongly locally p-connected at u, then

$$f(x) \geq f(u).$$

Proof: If possible let f(x)<f(u).

Since f is locally P-connected at u, therefore, it follows that

$$r_0(df)^+(u, H_{u,x^{(0+)}}) \leq 0 \quad (17)$$

with strict inequality if $r_\circ > 0$.

By the feasibility of x and $(u, r_\circ, r)$ for (P) and (D), respectively, we get

$$\sum_{j=1}^m r_j g_j(x) \leq \sum_{j=1}^m r_j g_j(u)$$

Now $\sum_{j=1}^m r_j g_j$ is strongly locally P-connected at u, so

$$d\left(\sum_{j=1}^m r_j g_j\right)^+ (u, H_{u,x^{(0+)}} \leq 0 \quad (18)$$

with strict inequality if some $r_j > 0, j = 1,2,..., m$.

Adding (17) and (18) and using (16), we get

$$r_\circ(df)^+(u, H_{u,x^{(0+)}}) + r^T(dg)^+(u, H_{u,x^{(0+)}}) < 0$$

which is the contradiction to (14).

Hence, $f(x) \geq f(u)$.

Theorem 3 (Strong Duality)

Let x* be an optimal solution of (P), $(df)^+(x^*, H_{x^*,x^{(0+)}})$ and $(dg_I)^+(x^*, H_{x^*,x^{(0+)}})$ be convex functions of x and $g_j, j \in J$, be continuous at x* with S convex or $S = R^n$. Then, there exist $r_0^* \in R, r^* \in R^m$, such that (x^*, r_0, r^*) is feasible for (D) and the values of the objective functions of (P) and (D) are equal at x*. Also, if for each feasible (u, r_0, r) for (D), f is locally P- connected and $\sum_{j=1}^m r_j g_j$ is strongly locally P-connected at u, then (x^*, r_0^*, r^*) is optimal for (D).

Proof: Since x* is an optimal solution of (P), therefore, by Theorem 1, there exist $r_0^* \in R, r^* \in R^m$ such that (x^*, r_0^*, r^*) is feasible for (D). Equality of objective functions for (P) and (D) follows trivially. Further, if (x^*, r_0, r^*) is not optimal for (D), then there exists (u, r_0, r), feasible for D, such that

$$f(u) > f(x^*)$$

which is which is a contradiction to weak –duality.

2. CONCLUSION

Weir and Mond provided a Fritz-John dual for the non-linear programming problems involving differentiable functions by using the Fritz-John optimality conditions instead of Kuhn-Tucker conditions and thus did not require a constraint qualification. In this paper we associated the Mond-Weir type Fritz-John dual to the non-linear programming problem.

3. REFERENCES

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