Modeling Curves via Fractal Interpolation with VSFF

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ABSTRACT
M. F. Barnsley proposed the concept of fractal interpolation function (FIF) using iterated function systems (IFS) to describe the real world objects. The purpose of this paper is to study the parameter identification method for FIF with vertical scaling factor functions (VSFF) for one dimensional data set and establish the generalized version of the analytic approach of Mazel [13].

Keywords
Iterated function system, Fractal interpolation function, Vertical scaling factor functions.

1. INTRODUCTION
Fractal geometry is of prime importance among the major recent developments in understanding the structures of natural objects. Barnsley [1] introduced the concept of fractal interpolation function (FIF) in 1986 using Hutchinson's operator [10] and iterated function systems. Thereafter, the theory of fractal interpolation has become a powerful tool in various branches of applied sciences and engineering. Barnsley [2], [3], Dalla and Drakopoulos [6], Manousopoulos et al [11]-[12], Mazel et al [13] and many others have extended and generalized it in diverse domains of activities. The applications and properties of such functions are further characterized by a number of papers reported in the literature, see for instance [3]-[5], [7], [9], [14]-[16] and several references thereof. The graphs of fractal interpolating functions can also be used to approximate image components of many natural objects such as the profiles of mountain ranges, the tops of clouds, stalactite-hung roofs of caves and horizons over forests. This technique is capable of achieving large amounts of data compression and thus it is widely used in simulation, modeling and computer graphics.

Usually vertical scaling factor is considered as a control parameter for the best fitting purpose in the theory of fractal interpolation. Mazel [13] and Manousopoulos [11], [12] described parameter identification method for modeling any discrete data set through fractal interpolation. In [2], Barnsley et al considered the case of constant vertical scaling factors for the purpose of fractal interpolation. But in actual practice, we may come across a number of data sets where the data points may not be scaled with constant vertical scaling factors. Thus there is a need of varying vertical scaling factors for approximating such type of data sets. Recently Feng et al. [8] studied fractal interpolation surfaces using function vertical scaling factors. Following them, we use vertical scaling factor functions (VSFF) to fit the data set and establish the generalized version of the analytic approach used by Mazel [13] for finding VSFF to construct IFS for FIF.

2. PRELIMINARIES
In this section we present the basic definitions and concepts required for our study.

Definition 1 [3]. Let (X, d) be a metric space. A transformation $w: X \to X$ is said to be Lipschitz with Lipschitz constant $s \in R$ iff $d(w(x), w(y)) \leq s\, d(x, y)$ for all $x, y \in X$. A transformation $f: X \to X$ is called contractive iff it is Lipschitz with Lipschitz constant $s \in [0, 1)$. A Lipschitz constant $s \in [0, 1)$ is also called a contraction factor.

Definition 2 [3]. A hyperbolic iterated function system (IFS) consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_n: X \to X$, with respective contractivity factors s_n for $n = 1, 2, ..., N$. This IFS is represented by $\{X; w_n: n = 1, 2, ..., N\}$ with contractivity factor $s = \max\{s_n: n = 1, 2, ..., N\}$.

Definition 3 [3]. Let (X, d) be a metric space and $H(X)$ the nonempty compact subsets of X. Then the Hausdorff metric h in $H(X)$ is defined as

$$h(A, B) = \max\{d(A, B), d(B, A)\} \text{ for all } A, B \in H(X),$$

where $d(A, B) = \max(\min(d(a, b): b \in B): a \in A)$.

We now state a Lemma of Barsnley [3] which guarantees a contraction map in $\{H(X), h\}$ out of a contraction map on (X, d).

Lemma 1 [3]. Let $w: X \to X$ be a contraction on a metric space (X, d) with contractivity factor s. Then $w: H(X) \to H(X)$ defined by

$$w(B) = \{w(x) : x \in B\} \quad \forall B \in H(X)$$

is a contraction on $\{H(X), h\}$ with contractivity factor s.

The following theorem of [2] ensures the existence of an attractor of an IFS.

Theorem 1 [2]. Let $\{X; w_n, n = 1, 2, ..., N\}$ be a hyperbolic iterated function system with contractivity factor s. Then the transformation W: $H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} w_n(B)$ for all $B \in H(X)$ is a contraction mapping on the complete metric space $(H(X), h)$ with contractivity factor s. That is, $h(W(B), W(C)) \leq s\, h(B, C)$ for all $B, C \in H(X)$. Its unique fixed point (or an attractor), $A \in H(X)$ obeys $A = W(A) = \bigcup_{n=1}^{N} w_n(A)$ and is given by $A = \lim_{n\to\infty} W^n(B)$ for any $B \in H(X)$.

3. FRACTAL INTERPOLATION FUNCTION WITH VERTICAL SCALING FACTOR FUNCTIONS
Let $\{(x_n, y_n) \in I\times R: n = 0, 1, \ldots, N\}$ be the data set, where $I = [x_0, x_N] \subset R$ and $x_0 < x_1 < \ldots < x_N$. The interpolation points divide I into N intervals $I_n = [x_{n-1}, x_n]$, $n = 1, 2, \ldots, N$. Now we define $w_n: I\times R \to R^2$, $n = 1, 2, ..., N$ in the following manner:

$$w_n\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}l_n(x)\\F_n(x,y)\end{pmatrix}=\begin{pmatrix}a_n x+b_n\\d_n(x)y+\varphi_n(x)\end{pmatrix} \qquad (1)$$

The constants are chosen such that each map w_n is constrained to map the endpoints of data set to the endpoints of the interval I_n. That is,

$$w_n\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix},\ w_n\begin{pmatrix}x_N\\y_N\end{pmatrix}=\begin{pmatrix}x_n\\y_n\end{pmatrix} \qquad (2)$$

for every n = 1, 2, 3, …, N. The real numbers a_n, b_n are completely determined by the interpolation points, while the $d_n(x)$ and $\varphi_n(x)$ are continuous function defined on I meeting Lipschitz condition for x. Also $d_n(x)$ i.e., the vertical scaling factor functions (VSFF), are satisfying the condition $|d_n(x)| < 1$ for $n = 1: N$ and all $x \in I$, so that the transformations w_n are contractive with respect to a metric equivalent to Euclidean metric.

Let $d=\max_n \sup_x |d_n(x)|$ and $M=\max_n \sup_x |\phi_n(x)|$. For any $h \geq M / (1\text{-}d)$ and $(x, y) \in I\times [-h, h]$, $|F(x, y)| \leq d\ h + M \leq h$. Then an iterated function system can be constructed as

$\{I\times [-h, h];\ w_n: n = 1: N\}$. (3)

Theorem 2. If $d < 1$ and $h \geq M/(1\text{-}d)$, then the IFS constructed above is hyperbolic in R^2 with respect to metric equivalent to Euclidian metric. Therefore, there exists a unique non empty set $A \subset I\times R$ such that

$$\bigcup_{n=1}^{N} w_n(A)=A,$$

where $d=\max_n \sup_x |d_n(x)|$ and $M=\max_n \sup_x |\phi_n(x)|$.

Proof. Since $d_n(x)$ and $\phi_n(x)$ for all n meet the Lipschitz condition for x, so there exist positive real numbers m_1, m_2 such that, for any $x \in I$,

$|d_n(x_1) - d_n(x_2)| \leq m_1|x_1 - x_2|$ and $|\phi_n(x_1) - \phi_n(x_2)| \leq m_2|x_1\text{-}x_2|$

Let $\rho : R^2 \times R^2 \rightarrow R$ such that

$\rho((x_1, y_1), (x_2, y_2)) = |x_1\text{-}\ x_2| + \theta\ |y_1 - y_2|$,

where θ is a parameter. It is obvious that ρ is a metric on the space R^2.

For any $(x_1, y_1), (x_2, y_2) \in I \times [-h, h]$,

$\rho(w_n((x_1, y_1), w_n(x_2, y_2)\ )$

$= \rho((a_n\ x_1+b_n,\ d_n(x_1)\ y_1 + \phi(x_1)), (a_n\ x_2+b_n,\ d_n(x_2)\ y_2 + \phi(x_2)))$

$\leq |a_n\ ||x_1\text{-}\ x_2| + \theta\ (|d_n(x_1)\ y_1 - d_n(x_2)\ y_2| + |\phi(x_1)-\phi(x_2)|)$

$\leq |a_n||x_1\text{-}\ x_2|+\theta(|d_n(x_1)|\ |y_1-y_2|+|y_2|\ |d_n(x_1)-d_n(x_2)|+ m_2|x_1\text{-}\ x_2|)$

$\leq |a_n\ ||x_1\text{-}\ x_2| + \theta\ (d\ |y_1-y_2|+h\ m_1\ |x_1-x_2|+ m_2|x_1\text{-}\ x_2|)$

$\leq (|a_n\ |+ (h\ m_1\ \theta + m_2\ \theta)\ |\ |x_1\text{-}\ x_2| + d\ \theta\ |y_1-y_2|$.

Let $\theta \leq (1- \max|a_n|)/2(h\ m_1+ m_2)$.

Then, we have

$\rho(w_n(x_1, y_1), w_n(x_2, y_2))$

$\leq \{(1+|a_n\ |)/2\}|x_1 - x_2|+ d\theta|y_1-y_2|$

$\leq \alpha_n\, \rho((x_1, y_1), (x_2, y_2))$,

where $\alpha_n = \max\{(1+|a_n|)/2,\ d\}$.

For any n = 1, 2, . . . , N, it is obvious that $0 \leq |\alpha_n| < 1$; then the w_n are all contraction mappings on $I \times [-h, h]$. It is said that the iterated function system (3) is hyperbolic for any $h \geq M /(1 - d)$. Therefore, there exists a unique non-empty compact set as the invariant set of the iterated function system (3).

Theorem 3. If $|d_n(x)| < 1$ for $n = 1: N$, $x \in I$ and $h \geq M /(1 - d)$ then the IFS defined in (3) with condition (2) has a unique invariant set A. The invariant set A is the graph of a continuous function f, which interpolates the data set i.e. $f(x_n) = y_n$ for $n = 0 : N$ and the graph of function f is invariant set of the IFS (3) if and only if f satisfies the equations

$f(l_n(x)) = F(x, f(x))$, for $x \in I$, where $n = 1 : N$.

3.1 Calculation for VSFF

Let $\{(x_n, y_n) \in I\times R$: n = 0, 1, …. , $N\}$ be the data set and (x_{n-1}, y_{n-1}) and (x_n, y_n) be two consecutive interpolating points. Also consider $\varphi_n(x)$, given in (1) is define as $\varphi_n(x) = c_n\ x + e_n$. Now equation (1) can be expressed as

$$w_n\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_n x+b_n\\d_n(x)y+c_n x+e_n\end{pmatrix}.$$

Clearly the function w_n maps the given data set of length $(x_N - x_0)$ to an interval of length $(x_n - x_{n-1})$.

Now, we apply least square approximation to calculate the VSFF for each interpolating data set. We find VSFF in three cases by considering three different types of functions:

1. First we assume a linear function expressed as $d_n(x) = u_n\ x + v_n$, $n = 1: N$. Then we find the expression for u_n and v_n for interval $[x_{n-1}, x_n]$ on applying the least square approximation as follows

$$u_n=\frac{\sum_{i=x_0}^{x_N} A_iC_i \times \sum_{i=x_0}^{x_N} B_i^2 - \sum_{i=x_0}^{x_N} A_iB_i \times \sum_{i=x_0}^{x_N} B_iC_i}{\sum_{i=x_0}^{x_N} B_i^2 \times \sum_{i=x_0}^{x_N} C_i^2 - \left(\sum_{i=x_0}^{x_N} B_iC_i\right)^2}$$

and

$$v_n=\frac{\sum_{i=x_0}^{x_N} A_iB_i \times \sum_{i=x_0}^{x_N} C_i^2 - \sum_{i=x_0}^{x_N} A_iC_i \times \sum_{i=x_0}^{x_N} B_iC_i}{\sum_{i=x_0}^{x_N} B_i^2 \times \sum_{i=x_0}^{x_N} C_i^2 - \left(\sum_{i=x_0}^{x_N} B_iC_i\right)^2},$$

where

$$A_i=\frac{x_n-x_{n-1}}{x_N-x_0}x_i+\frac{x_N y_{n-1}-x_0 y_n}{x_N-x_0}-y_m$$

$$B_i=(x_i-1)\frac{y_N-y_i}{x_N-x_0}-y_i$$

$$C_i=(x_i-1)\frac{x_N y_N-x_0 y_i}{x_N-x_0}-x_i\, y_i$$

and chose m such that $x_m \approx a\cdot x_i + e$.

2. Again we have a linear function expressed as $d_n(x) = u\ x + v$ and constrained by $d_n(x_0) = s_{n-1}$ and $d_n(x_N) = s_n$, i.e.

$d_n(x) = s_{n-1}+(s_n - s_{n-1})(x - x_0)/(x_N - x_0)$, for $n = 1{:}\ N$.

Then, choose $0 \le s_0 < 1$, and apply least square approximation to find s_n, $n = 1{:}\ N$. we get

$$s_n = \frac{\sum_{i=x_0}^{x_N} A_i B_i}{\sum_{i=x_0}^{x_N} B_i^2},$$

where

$$A_i = \frac{x_i y_i + x_0 y_N - x_i y_N - x_0 y_i}{x_N - x_0}$$

$$B_i = \frac{(y_n - y_{n-1})x_i + s_{i-1}(x_N y_i + x_i y_0 - x_N y_0 - x_i y_i) + x_N y_{n-1} - x_0 y_n}{x_N - x_0} - y_m$$

and choose m such that $x_m \approx a \cdot x_i + e$.

3. A quadratic function $d_n(x) = -x^2 + u\ x + v$ constrained by $d_n(x_0) = s_{n-1}$ and $d_n(x_N) = s_n$, i.e.

$$d_n(x) = s_{n-1} + \frac{(s_n - s_{n-1})(x - x_0)}{(x_N - x_0)} - x_0 x_N + (x_N + x_0)x - x^2$$

for $n = 1{:}\ N$. Then, choose $0 \le s_0 < 1$, and apply least square approximation to find s_n, $n = 1{:}\ N$. we get

$$s_n = \frac{\sum_{i=x_0}^{x_N} A_i B_i}{\sum_{i=x_0}^{x_N} B_i^2},$$

where

$$A_i = \frac{x_i y_i + x_0 y_N - x_i y_N - x_0 y_i}{x_N - x_0}$$

$$B_i = \frac{(y_n - y_{n-1})x_i + s_{i-1}(x_N y_i + x_i y_0 - x_N y_0 - x_i y_i) + x_N y_{n-1} - x_0 y_n + \left(x_i x_N + x_i x_0 - x_0 x_N - x_i^2\right)}{x_N - x_0} - y_m$$

and choose m such that $x_m \approx a \cdot x_i + e$.

3.2 Algorithm to select the best interpolation interval

To select the best interpolation interval I_n and corresponding VSFF i.e. $d_n(x)$ for the given data set, we present the following algorithm.

a. Choose the initial point on the function H as the first interpolation point and the left endpoint of the first section.

b. Choose the next point on the function as the next interpolation point and the right endpoint for that section.

c. Calculate the contraction factor for $d_n(x)$ associated with the interpolation section defined by interpolation points.

d. If $|d_n(x)| < 1$ go to step v, otherwise go to step ii

e. Compute the map parameters with end point condition (2) and form the map w_n associated with the pair of interpolation points. Apply the map to each point of the function to yield $w_n(H)$.

f. Compute and temporarily store the distance between the original function located between the pair of interpolation points, say H, and $w_n(H)$.

g. Repeat steps ii-vi until the end of the function is reached.

h. Store the pair of interpolation points and contraction factor which yield the minimum value of $h(H_i, w(H))$ from steps v and vi.

i. Let the right endpoint of the stored pair of interpolation points be the left endpoint of the next pair of interpolation points.

j. Go to step ii and continue until the entire function has been searched.

3.3 Example

We consider an image of the profile of a mountain range for our computation purpose. For applying fractal interpolation technique the image is digitized with 1233 points. The vertical scaling factor functions are computed for the cases given in

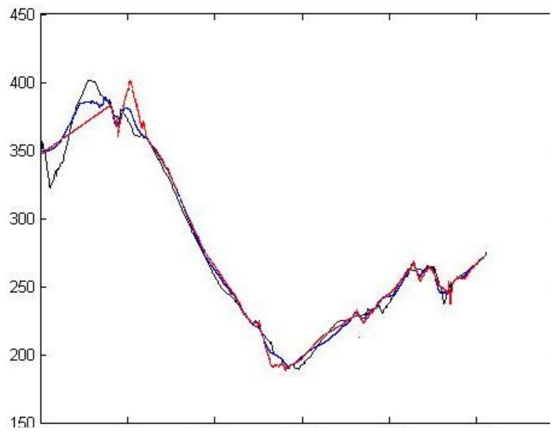

Fig 1. Original and fractal interpolated curves

---- Original
---- FIF with FVSF (L)
.... FIF with FVSF (Q)

section 3.1. Using algorithm 3.2, we calculate the set of interpolating data points for the best fitting. The best fitted curves are plotted in Figure 1 for linear and quadratic cases. The black curve shows the original curve consisting of 1233 points while red and blue curves are the fractal interpolated curves drawn by taking only 13 interpolating points by considering linear VSFF and quadratic VSFF respectively.

4. CONCLUSIONS

The modeling of the various natural or other problems require some degree of flexibility and smoothness for best fitting purpose. Our approach of fractal interpolation function with vertical scaling factor functions used in this paper ensures the achievement of this extra flexibility and smoothness.

5. REFERENCES

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