Study on Optimization Problem of Propellant Mass Distribution under Restrictive Conditions in Multistage Rocket

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ABSTRACT

When a mission for multistage rocket is fixed, the problem of optimum staging for rocket design comes in. In the proposed study we have discussed the treatment of optimization problem under two subsidiary conditions of restriction, namely for a constant initial gross mass of the rocket, to find optimum stage-mass distribution to achieve minimum propellant mass for a given required payload and burnout velocity. The usual assumption of drag free and gravitation less environment is taken.

Keywords
Propellant factors, Exhaust velocities and optimum stage mass distribution of rocket, Lagrange's multipliers.

1. INTRODUCTION

There are four forces which move the rocket through the air. These are lift, drag, weight and thrust. Lift and drag forces are generated and act on a model rocket as it flies through the air which are called aerodynamic forces. Lift acts perpendicular to the direction of motion through the air. Drag acts in the same direction as the relative air flow. The lift and drag act through the centre of pressure of the rocket and cause the rocket to rotate about the centre of gravity in flight.

Weight is the force generated by the gravitational attraction of the earth on the rocket, directed towards the centre of earth. The magnitude of this force depends on the mass of all of the parts of the rocket itself, amount of fuel and payload.

The thrust force of a rocket is the reaction experienced by its structure due to the ejection of high velocity matter. Thrust is generated by the engine of the rocket through the application of Newton's third law of motion: "action and reaction". A gas or working fuel is accelerated by engine nozzle and the reaction to this acceleration produces a force on the engine. Thrust is normally directed forward along the center-line of the aircraft.

In rocket vehicle with three stages the first stage propels the rocket vehicle from the launch pad. When the first stage has burned all of its fuel, it is jettisoned and the engine in the second stage propels the remainder of the vehicle to a higher altitude and velocity. After the second stage exhausts its fuel supply it is also jettisoned and the third stage engine is fired to accelerate the spacecraft to the velocity and altitude prescribed for mission.

A multistage rocket jettisons its stages as they become useless for further propulsion. As a result more it provides acceleration for a given amount of fuel. A rocket having two or more engines stacked one on top of another and firing in succession is called multistage. When the mass ratios of the stages are equal, the payload is a maximum for gravity-free vacuum flight and the distribution of masses between the stages is optimum.

Here we assume that all outside forces on the rocket such as gravity and aerodynamic drag are so small in comparison with the thrust of the rocket engine that these outside forces can be neglected. We also assume that the exhaust gases are expelled straight out of the back of the rocket at a constant speed c relative to the rocket. We have no need to know about the design of rocket to study our optimization problem.

Consider that at initial time t the rocket is moving with velocity *v* and mass *M.* Principle of Conservation of Linear Momentum yields the Rocket Equation [Zar Htun Kyi, Khin. and Myint, Ohnmar (2008)].

$$\frac{dv}{dt} = -\frac{c}{M}\frac{dM}{dt}.$$

The rocket equation relates the acceleration $\frac{dv}{dt}$ of the rocket vehicle to the rate $\frac{dM}{dt}$ of fuel consumption when outside forces acting on the rocket are neglected.

Where the minus sign indicates that the rocket moves forward since the rocket loses mass as it accelerates, therefore $\frac{dM}{dt}$ will be negative.

Now consider a rocket vehicle with n stages. The *i*th stage is to have total mass M_i, propellant factor S_i and engine exhaust speed c_i , our objective is to choose M_i for $i = 1, 2, ...n$ to minimize the total gross mass of the rocket.

Let M_o is the initial gross mass of the rocket such that

$$M_o = M_1 + M_2 + ...M_n,$$

subject to the constraint that the final velocity of the rocket after burnout of the last stage is a prescribed value *v.*

Goldsmith (1957), Hall and Zambelli (1958), Sohurmann (1957), Subotowicz (1957) have considered optimization of the gross weight of the rocket for given required payload and burnout velocity. Coleman (1961) has discussed the optimum stage weight distribution for optimum payload ratio under given burnout velocity. The results of earlier investigations of Weisbord (1958), Subotowicz (1958), Hall and Zambelli (1958) and Malina and Summerfield (1947) are deduced as particular cases of the velocity constraint, optimization problem.

Let P is the mass of the payload in multistage rocket of n stages so that

$$M_o = P + \sum_{i=1}^{n} M_i \,, \tag{1}$$

where $i = 1, 2, ..., n$ and M_i is the mass of i^{th} stage.

Then the ratio of the total propellant mass to the payload P is given by

$$L = \frac{\sum_{i=1}^{n} M_{P_i}}{P} \,, \tag{2}$$

where M_{P_i} is propellant mass of the i^{th} stage.

The propellant factor S_i, corresponding to i^{th} stage is given by the relation

$$S_i = \frac{M_i}{M_{P_i}} \,. \tag{3}$$

With the help of (1) and (3) we have

$$P = M_o - \sum_{i=1}^{n} S_i M_{P_i} \,. \tag{4}$$

With the help of the Rocket equation the burnout velocity can be given by

$$V = \sum_{i=1}^{n} c_i \log\left(\frac{M_o}{M_o - M_{P_i}} \right), \tag{5}$$

where M_o is the gross mass of the rocket after $(i-1)^{th}$ stage and c_i is the exhaust velocity of i^{th} stage.

2. SIMPLIFICATION OF THEORETICAL EQUATIONS

Using equation (4) in equation (5), the burnout velocity

$$V = \sum_{i=1}^{n} c_i \log\left(\frac{\sum_{r=i}^{n} S_r M_{P_r} + P}{\sum_{r=i}^{n} S_r M_{P_r} + P - M_{P_i}} \right)$$

$$= \sum_{i=1}^{n} \log\left(\frac{\sum_{r=i}^{n} S_r M_{P_r} + P}{\sum_{r=i}^{n} S_r M_{P_r} + P - M_{P_i}} \right)^{c_i}$$

$$\text{or } e^V = \prod_{i=1}^{n} \left(\frac{\sum_{r=i}^{n} S_r M_{P_r} + P}{\sum_{r=i}^{n} S_r M_{P_r} + P - M_{P_i}} \right)^{c_i} \tag{6}$$

Now define a function

$$\phi = e^V \tag{7}$$

The knowledge and optimization of V will give the knowledge and optimization of ϕ.

The problem is now reduced to optimize propellant mass ratio L under restrictive conditions of P and ϕ using λ_1 and λ_2 as Lagrange's multipliers, the optimization equations are

$$\frac{\partial L}{\partial M_{P_1}} + \lambda_1 \frac{\partial P}{\partial M_{P_1}} + \lambda_2 \frac{\partial \phi}{\partial M_{P_1}} = 0 \,, \tag{i}$$

$$\frac{\partial L}{\partial M_{P_2}} + \lambda_1 \frac{\partial P}{\partial M_{P_2}} + \lambda_2 \frac{\partial \phi}{\partial M_{P_2}} = 0 \,, \tag{ii}$$

$$\frac{\partial L}{\partial M_{P_3}} + \lambda_1 \frac{\partial P}{\partial M_{P_3}} + \lambda_2 \frac{\partial \phi}{\partial M_{P_3}} = 0 \,, \tag{iii}$$

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$$\frac{\partial L}{\partial M_{P_n}} + \lambda_1 \frac{\partial P}{\partial M_{P_n}} + \lambda_2 \frac{\partial \phi}{\partial M_{P_n}} = 0 \,. \tag{8}$$

And for given values of P and V

$$P - M_o - \sum_{i=1}^{n} S_i M_{P_i} = 0 \,,$$

$$\phi - e^V = 0 \,.$$

By these $(n+2)$ equations we can solve our optimization problem.

3. SIMPLIFICATION OF OPTIMIZATION EQUATIONS

From equation (2) we have

$$\frac{\partial L}{\partial M_{P_i}} = \frac{1}{P}, i = 1, 2, \ldots\ldots\ldots, n. \qquad (9)$$

Using equation (9) in (8) and applying $(i)-(ii)$ and $(ii)-(iii)$, we have

$$\lambda_1\left[\frac{\partial P}{\partial M_{P_1}} - \frac{\partial P}{\partial M_{P_2}}\right] = -\lambda_2\left[\frac{\partial \phi}{\partial M_{P_1}} - \frac{\partial \phi}{\partial M_{P_2}}\right],$$

$$\lambda_1\left[\frac{\partial P}{\partial M_{P_2}} - \frac{\partial P}{\partial M_{P_3}}\right] = -\lambda_2\left[\frac{\partial \phi}{\partial M_{P_2}} - \frac{\partial \phi}{\partial M_{P_3}}\right].$$

By the elimination of Lagrange's multipliers λ_1 and λ_2 the above two equations become

$$\left(\frac{\partial P}{\partial M_{P_1}} - \frac{\partial P}{\partial M_{P_2}}\right)\left(\frac{\partial \phi}{\partial M_{P_2}} - \frac{\partial \phi}{\partial M_{P_3}}\right) = \left(\frac{\partial P}{\partial M_{P_2}} - \frac{\partial P}{\partial M_{P_3}}\right)\left(\frac{\partial \phi}{\partial M_{P_1}} - \frac{\partial \phi}{\partial M_{P_2}}\right)$$

(10)

The equation (10) in general can be written as

$$\left(\frac{\partial P}{\partial M_{P_i}} - \frac{\partial P}{\partial M_{P_{i+1}}}\right)\left(\frac{\partial \phi}{\partial M_{P_{i+1}}} - \frac{\partial \phi}{\partial M_{P_{i+2}}}\right) = \left(\frac{\partial P}{\partial M_{P_{i+1}}} - \frac{\partial P}{\partial M_{P_{i+2}}}\right)\left(\frac{\partial \phi}{\partial M_{P_i}} - \frac{\partial \phi}{\partial M_{P_{i+1}}}\right)$$

(11)

where $i = 1, 2, \ldots\ldots\ldots, (n-2)$.

Now from equation (4)

$$\frac{\partial P}{\partial M_{P_i}} = -\sum_{i=1}^{n} S_i. \qquad (12)$$

Using equation (12) the equation (11) becomes

$$\left(S_i - S_{i+1}\right)\left(\frac{\partial \phi}{\partial M_{P_{i+1}}} - \frac{\partial \phi}{\partial M_{P_{i+2}}}\right) = \left(S_{i+1} - S_{i+2}\right)\left(\frac{\partial \phi}{\partial M_{P_i}} - \frac{\partial \phi}{\partial M_{P_{i+1}}}\right) \qquad (13)$$

and $P - M_o - \sum_{i=1}^{n} S_i M_{P_i} = 0$, (14)

$$\phi - e^V = 0. \qquad (15)$$

The above n equations give the n values of M_{P_i} for a given set of propellant factors and exhaust velocities. Then from equation (3), optimum stage mass distribution can be found by iterative method.

4. CONCLUSION

If we consider a three stage rocket design the equations (13) (14) and (15) for a given set of values of propellant factor and exhaust velocities, constitute three algebraic equations in the unknowns M_{P_1}, M_{P_2} and M_{P_3} where gross mass of the rocket, payload and burnout velocity have required fixed values. The equations can be numerically solved by iterative methods. Having known the optimum values of M_{P_1}, M_{P_2} and M_{P_3} ,optimum stage mass distribution can be determined.

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