Linear Quadratic Gaussian Control for Two Interacting Conical Tank Process

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ABSTRACT
In this paper optimal control for two interacting conical tank process (TICTP) was designed. The optimal control is obtained by LQG solution with optimal kalman filter. This paper describes the theatrical base and practical application of an optimal dynamic regulator using model based Linear Quadratic Gaussian (LQG) control design for nonlinear process. This LQG regulator consists of an optimal state-feedback controller and an optimal state estimator. In this case, a performance criterion is minimized in order to maintain level of the water in both tanks.

Keywords
Linear quadratic Gaussian controller; Kalman estimator; Two interacting conical Tank Process; MIMO nonlinear process.

1. INTRODUCTION
Optimal control theory is emerging technique in advanced process control. In optimal control theory, the calculus variations and optimization techniques combined used to form path of controlled variable such that cost function like control energy and state fluctuation is minimized[1][2][3]. In general Model predictive control strategy is used mostly on higher level , the performance improvement of lower level PID loops gives improved performance in multivariable systems. But Linear quadratic Gaussian control provides optimal and robust control strategy to Multivariable process[4][5]. Linear Quadratic Gaussian design is a optimal control theory which has many application in control engineering problem. This LQG technique widely used in Medical process controllers, in nuclear power plants and motor control systems. In Linear Quadratic Regulator(LQR) model, the resulting control law are linear with respect to state variable. The control law is easy to compute but the disadvantage in LQR is, it fails to perform well when the system is subjected to disturbances [6]. So the special case of controller is designed using separation principle [7]. The concept of Linear Quadratic regulators and Kalman estimator combined to form LQG control model for better servo and regulatory response. Linear Quadratic Regulator is optimal robust controller to disturbance rejection but for servo problem it may not track set point due to some model mismatch and error in the sensor output. So a Kalman filter used to minimize the asymptotic covariance of the estimation of error when the process encounter with disturbances [8][9][10].The state feedback gain matrix uses state as input from Kalman estimator not from real process. The optimal state feedback gain and optimal state estimator gain determined to form optimal LQG controller. The state feedback LQ-optimal gain is determined by Ricatti equation [11][12]. The main advantage of LQG control is that closed loop stability is guaranteed when the system is accurate. Cylindrical tanks used in many process industry for discharge of liquid. To control liquid level and flow in process tanks are still challenging problem in process industries.

The paper is organized as follows. Section II presents the two interacting conical tank process modeling. LQG Controller design described in section III,IV. Simulation results and controller Performance analysis are shown and discussed in section V. Final conclusions are given in section VI.

2. TWO INTERACTING CONICAL TANK PROCESS DESCRIPTION

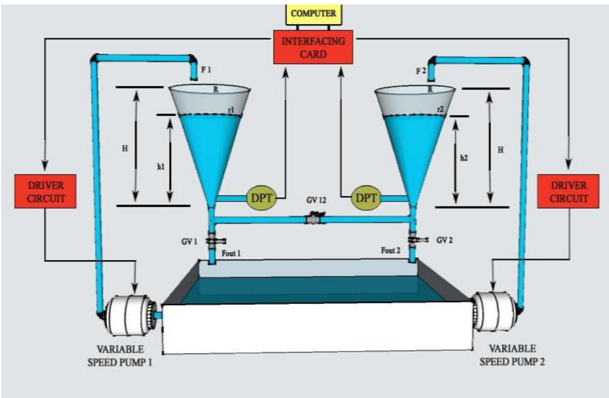

Fig 1: Schematic diagram of two interacting conical tank

The proposed system consists of two conical tanks which are in the shape of an inverted cone fabricated from a sheet metal. The height of the process tank is 50cm and the top end, tapering end diameters are 40cm and 14cm. The two tanks are connected through an interacting pipe with valve (HV1). The interaction of process can be changed by position of this valve (HV1). It has a reservoir to store water and this is supplied through the pumps to the tanks. Provisions for water inflow and outflow are provided at the top and bottom of the tank respectively. Gate valves, one at the outflow of the tank1 and the other at the outflow of the tank2 are connected to maintain the level of water in the tanks. Variable Speed pump work as actuator and it is used to discharge the water from reservoir tank to process tanks. The speed of pump directly propositional to the input voltage. It consists of differential pressure transmitter for measuring the bottom pressure created by water level and it gives height in terms of milliamps.

2.1. Mathematical Modelling

$$\frac{dh_1}{dt} = \frac{K_{pp1} U_1 - \beta_1 a_1 \sqrt{2gh_1} - sign(h_1 - h_2)\beta_{12} a_{12} \sqrt{2g\left|h_1 - h_2\right|}}{\pi R_1^2 \frac{h_1^2}{H_1}} \quad (1)$$

$$\frac{dh_2}{dt} = \frac{K_{pp2} U_2 - \beta_2 a_2 \sqrt{2gh_2} + sign(h_1 - h_2)\beta_{12} a_{12} \sqrt{2g\left|h_1 - h_2\right|}}{\pi R_2^2 \frac{h_2^2}{H_2}} \quad (2)$$

The nominal values of the parameters and variables are tabulated in table 1.

Table 1. Nominal values of the parameters used

Parameter	Description	Value
R	Top Radius of conical tank	20 cm
H	Maximum height of Tank1&2	50 cm
U_1, U_2	Input Voltage to pump1 & 2	0-10 V
K_{pp1}, K_{pp2}	Pump gain	75 $cm^3/V.Sec$
β_1	Valve Co-efficient of MV1	50 cm^2/s
β_{12}	Valve co-efficient of Mv12	35cm^2/s
β_2	Valve co-efficient of Mv2	50cm^2/s
a_1, a_{12}, a_2	Cross section Area of pipe	1.2272 cm^2

2.2. Open Loop data

The open loop data was generated in the conical tank system by varying the inflow rate F_1 in tank 1 and noting down the respective level h1 and h2 and is tabulated in Table 2. The I/O characteristics and linearized region of the two conical tank process is shown (see Figure 2).

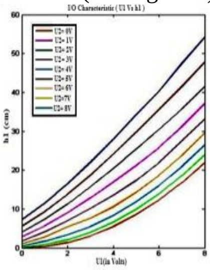

Fig 2a: U_1 Vs h1

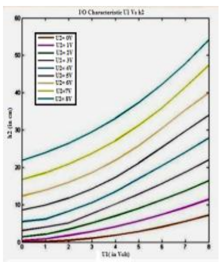

Fig.2b: U_1 Vs h2

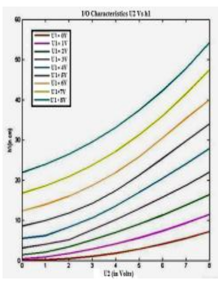

Fig2c: U_2 Vs h1

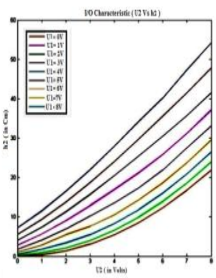

Fig.2d: U_2 Vs h2

Fig 2: I/O characteristics of the conical tank process

Thus the piecewise Linearization method is used to separate the whole non-linear region into various regions. The characteristics are divided into different region. The operating points are found out for each region.

Table 2. Operating conditions and the conventional state space and transfer function model of the interacting conical tank process

Region	Operating points	State space	Transfer function matrix G(s)
I h_1=0-5 h_2=0-4	U_{1s}=2.2 U_{2s}=1.1 h_{1s}=2.5 h_{2s}=2.1	$A=\begin{bmatrix}-6.9 & 4.3\\ 6.4 & -10.8\end{bmatrix}$ $B=\begin{bmatrix}7.7 & 0\\ 0 & 11.5\end{bmatrix}$ $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{7.704s+83.08}{s^2+17.73s+47.07} & \frac{49.64}{s^2+17.73s+47.07}\\ \frac{49.64}{s^2+17.73s+47.07} & \frac{11.51s+79.89}{s^2+17.73s+47.07}\end{bmatrix}$
II h_1=5-15 h_2=4-12	U_{1s} =4.5 U_{2s} =2.1 h_{1s}=10.1 h_{2s}=8.1	$A=\begin{bmatrix}-0.22 & 0.13\\ 0.21 & -0.35\end{bmatrix}$ $B=\begin{bmatrix}0.49 & 0\\ 0 & 0.75\end{bmatrix}$ $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{0.49s+0.17}{s^2+0.568s+0.048} & \frac{0.10}{s^2+0.568s+0.048}\\ \frac{0.10}{s^2+0.568s+0.0486} & \frac{0.75s+0.16}{s^2+0.568s+0.048}\end{bmatrix}$
III h_1=15-25 h_2=12-20	U_{1s}=7 U_{2s} =2 h_{1s}=20.9 h_{2s}=13.8	$A=\begin{bmatrix}-0.02 & 0.016\\ 0.03 & -0.07\end{bmatrix}$ $B=\begin{bmatrix}0.11 & 0\\ 0 & 0.26\end{bmatrix}$ $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{0.11s+0.008}{s^2+0.105+0.0016} & \frac{0.004}{s^2+0.105+0.0016}\\ \frac{0.004}{s^2+0.105+0.0016} & \frac{0.26\ s+0.007}{s^2+0.105+0.0016}\end{bmatrix}$
IV h1=25-45 h2=20-45	U_{1s}=7.6 U_{2s} =5 h_{1s}=35.1 h_{2s}=32.6	$A=\begin{bmatrix}-0.01 & 0.009\\ 0.01 & -0.016\end{bmatrix}$ $B=\begin{bmatrix}0.04 & 0\\ 0 & 0.04\end{bmatrix}$ $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{0.04s+0.0006}{s^2+0.02s+9.91e^{-05}} & \frac{0.0004}{s^2+0.02s+9.91e^{-05}}\\ \frac{0.0004}{s^2+0.02s+9.9e^{-05}} & \frac{0.048s+0.0006}{s^2+0.028s+9.9e^{-05}}\end{bmatrix}$

3. MULTI LOOP CONTROLLER DESIGN

The most attractive advantages of multiloop control system is simple controller structure and easiness to handle loop failure. The Variable pairing is selected by Relative gain array method. The influence between inputs on outputs is calculated and the high influence input and output pair is chosen for control. The input1 (voltage applied to pump1) is paired with output1 (height of tank1) and input2 (voltage applied to pump1) is paired with output2 (height of tank2).

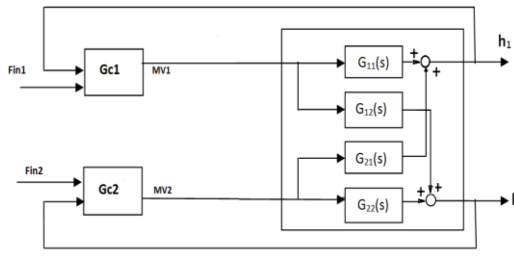

Fig 3: Multiloop Control Scheme

PI controller transfer function is $Gc(s)=\left(\frac{k_p s+k_i}{s}\right)$

The second order transfer function is reduced into first order plus dead time and diagonal transfer function is tuned using Ziegler Nichols method.

Table.3 CDM based PI and ZN based PI values for TICTP

Conical Transfer Function FOPDT	ZN based PI	
	Loop1	Loop2
$\begin{bmatrix} \frac{5.227e^{-1s}}{51.22\,S+1} & \frac{2.576\,e^{-1s}}{66.29\,S+1} \\ \frac{2.576\,e^{-1s}}{66.29\,S+1} & \frac{4.727\,e^{-1s}}{24.65\,S+1} \end{bmatrix}$	K_p=8.827 K_i=2.758	K_p=4.711 K_i=1.535

4. LINEAR QUADRATIC GAUSSIAN CONTROLLER DESIGN

Linear-quadratic-Gaussian (LQG) control is a modern state-space technique which uses a state-space model of the plant. The model of process will not be a perfect replica of real process. The real time system is corrupted by noise and disturbance. The state space model of system which is affected by process noise 'w' and measurement noise 'v' is given below,

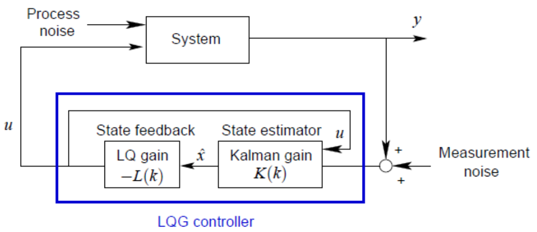

Fig4 : Block diagram of LQG Regulator

$$\dot{x} = Ax + Bu + Gw \tag{3}$$
$$y = Cx + Du + Hw + v \tag{4}$$

Assume *w* and *v* are white noises.LQG regulator consist of kalman state estimator and optimal feedback gain. The rank of controllability and observability matrix of system is same as

number of state of process. So the system is fully observable and controllable. Estimator can be used for estimate all the states.

The performance of LQG control is measured by performance index,

$$J(u) = \int_0^\infty \{x^T Qx + u^T Ru\}dt \tag{5}$$

Minimum values of J(u) represents minimum effect of controller energy 'u' and minimization of state variable changes in process. The selection of Q and R matrix is logical and meaningful. The Q and R matrixes selected for minimum value of performance index j(u) which means that the minimization of square of manipulated input and minimization of square of state variables fluctuations. In MIMO process, Influences between each input and output variables varies. The weightage for each manipulated variable varies based on the impact of output variables. The weighting matrices Q,R are fixed for minimizing the J(u). Q,R are the controller design parameters , large Q penalizes transients of x , large R penalizes usage of control action 'u'.

The state feedback gain is fixed for the minimum value of J(u). This minimization of gain matrix 'K' find by solving Riccati equation[11][12]. The regulator response of this linear quadratic regulator will be never affected by disturbance. But the setpoint tracking is not possible in LQR controller. The law of LQR controller is "u=-kx" that reduce the state fluctuation. Normally this gain is called state feedback gain or LQ-optimal gain. Control action is based on the state of process, if there is error in process state measurement due to model mismatch and noise, then controller will take action only for error measurement. To avoid this issue, kalman estimator is used to estimate the state of process.To form the LQG regulator, simply connect the Kalman filter and LQ-optimal gain K as shown (see Figure 5).

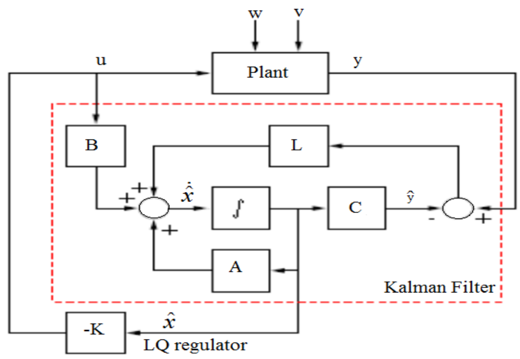

Fig 5: Linear quadratic Gaussian controller

State cannot be fully measurable in real time system, so the states are estimated based on the output. This regulator has state-space equation is

$$\dot{\hat{x}} = A\hat{x} + Bu \tag{6}$$

Estate estimated error $e = x - \hat{x}$,

Dynamic equation of error is

$$\dot{e} = Ax - A\hat{x} = Ae \tag{7}$$

The error goes to zero asymptotically when the State matrix A is stable. If the state matrix A is unstable then error become uncontrollable and estimated states $\hat{x}$ grows further apart from process state x. So correction term L is introduced to provide direct corrective action to minimize the model error.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{8}$$

where $\hat{x}(0) = 0$

$$\hat{y} = C\hat{x} \tag{9}$$

But in real time process, states are generally corrupted by 'w' is process noise and 'v' is measurement noise. State space model of noise corrupted system is model in equation 6,7. Now error dynamic of observer changes due to corrupted white noise.

$$\begin{aligned} \dot{e} &= (Ax + Bu + w) - (Ax + Bu + L(Cx + v - C\hat{x}) \\ \dot{e} &= (A - LC)e + w - Lv \end{aligned} \tag{10}$$

The error estimation will not go to zero because of process noise 'w' and measurement noise 'v'. So LQG technique, the output-feedback problem reduced by the estimated state. Assume the measurement noise v and process noise w are uncorrelated zero mean Gaussian white noise. Kalman filter is robust filter and very good optimal estimator when the system affected by Gaussian white noise. Specifically, it minimizes the asymptotic covariance of the estimation error $x - \hat{x}$.

$$\lim_{t \to \infty} E\left((x - \hat{x})(x - \hat{x})^T\right) \tag{11}$$

The optimal Kalman gain L minimizes $E\|e(t)\|^2$ is

$L = PC^{*}V^{-1}$. where P is the unique positive- semi definite solution of the ARE

$$PA^{*} + AP - PC^{*}V^{-1}CP + W = 0 \tag{12}$$

Kalman filter is equivalent to designing an LQR controller on the dual system (A*,C*,B*,D*) with Q = W, R = V. The system is fully observable and controllable. So, this estimated state given to the kalman gain L.

5. DESIGN OF LQG CONTROLLER FOR TICTP

Steps for Designing Linear Quadratic Gaussian Regulatory,

i. Check for controllability and observability of the system.
ii. Choose Q and R such that $Q = M^TM$, with (A,M) detectable, and $R = R^T > 0$
iii. This equation is the matrix algebraic Riccati equation (MARE), whose solution P is needed to compute the optimal feedback gain K.
iv. Solve the Riccati equation $PA + A^TP + Q - PBR^{-1}B^TP = 0$, And compute $K = R^{-1}B^TP$,Simulate the initial response of $\dot{x} = (A + BF)x$ for different initial conditions.
v. If the transient response specifications and/or the magnitude constraints are not met, and again step 1 is considered to re-choose the value of Q and R.

5.1. Checking of Controllability and Observability

The controllability of matrix depends on A and B matrix and observability of system depends on matrix A and C matrix. The system is said to be fully observable if all the states of system can be externally measured.

A system is controllable if condition W is satisfied

$$W: \text{rank}\,[B\ AB\ \ldots\ldots A^{n-1}B] = n \tag{13}$$

Therefore the value of

$$W = \begin{bmatrix} 0.114 & 0 & -0.003 & 0.004 \\ 0 & 0.26 & 0.004 & -0.020 \end{bmatrix}$$

And rank (W) = 2, Finally the system satisfies the condition W, therefore the system is controllable.

A system is observable if condition M is satisfied

$$M: \text{rank} \begin{bmatrix} C \\ CA \\ . \\ CA^{n-1} \end{bmatrix} = n \tag{14}$$

Therefore the value of

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.0298 & 0.0162 \\ 0.0375 & -0.0760 \end{bmatrix}$$

And rank (M) = 2; The system satisfies the condition M, therefore the system is observable.

Here the system is both controllable and observable. Therefore the design of LQG controller is possible. Since the third region of TICTP is considered for the design of LQG controller. Similarly it will be designed for rest of the region. The state space for third region is

$$A = \begin{bmatrix} -0.0298 & 0.0162 \\ 0.0375 & -0.0760 \end{bmatrix}; B = \begin{bmatrix} 0.1140 & 0 \\ 0 & 0.2629 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

$Q_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; Equal weightage given to state h1 and h2 of the tank. And $R_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ fixed for minimizing the controller energy and state variation.

$$Q_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; R_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

The row size of QN specifies the length of w and NN is set to 0 when omitted. By means of kalman filter, the estimated states are found

$$\hat{A} = \begin{bmatrix} -0.119 & -0.0005 \\ 0.0208 & -0.275 \end{bmatrix}$$
$$\hat{B} = \begin{bmatrix} 0.089 & 0.0167 \\ 0.0167 & 0.1994 \end{bmatrix}$$
$$\hat{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \hat{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

Kalman returns the estimator gain L and the steady-state error covariance P (solution of the associated Riccati equation).

$$L = PC^TR^{-1}$$
$$0 = AP + PA^T - PC^TR^{-1}CP + Q, P \geq 0, \tag{18}$$
$$Q = E(ww^T), R = E(vv^T)$$

From the above equation A,B,C,D,R,Q can be substituted and can find the value of P, With the help of all those values, the state feedback 'L' can be easily found and can be used for designing an LQG controller for TICTP.

$$P = \begin{bmatrix} 0.0891 & 0.0167 \\ 0.0167 & 0.1994 \end{bmatrix}$$

The state feedback gain will be

$$L = \begin{bmatrix} 0.089 & 0.017 \\ 0.017 & 0.199 \end{bmatrix}$$

6. ANALYSIS

Two interacting conical tank system was analyzed by developing simulation model. The interaction of this process is highly nonlinear. So piecewise linearization method is used to separate the whole non-linear region into various linear regions. The characteristics are divided into different region. The operating points are found out for each region. Linear state space format is obtained for four different regions by using jacobian matrix method.

The combination of state estimator and state feedback LQ gain will be LQG controller for TICTP. Thus the calculated values are used for obtaining the proper LQG controller which helps to reject disturbance and to track set point accordingly. According to the process interaction, Q and R values are chosen. Since in TICTP, the interaction is equal Q and R matrix chosen as [1 0; 0 1] and respective gain value is used in state feedback. For third linear region Linear Quadratic Gaussian regulatory was designed.

The response of disturbance rejection at steady sate is shown (see Figure 6) and performance validation between LQG and PI is tabulated in table.4. The disturbance is applied to study the regulatory performance. At 300sec and 400sec positive disturbance is applied to both tanks. The disturbance is rejected quickly by LQG controller.

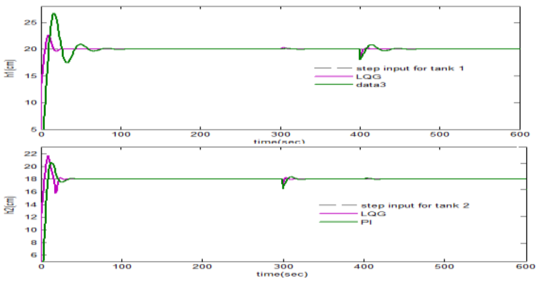

Fig 6: Regulatory response of LQG controller

Table.4 Performance Indices for Regulatory Response of LQG and PI controller

Control Strategy		IAE	ISE	ITAE
PI	Loop1	237.1	1816	9772
	Loop2	113.4	1024	3349
LQG	Loop1	146.37	1173	2683
	Loop2	81.21	877.5	1395

The servo and regulatory response of LQG and PI controller is shown (see Figure 4). The small overshoot can be eliminated by proper selection of integral gain. For this simulation response integral gains chosen by trial and error method. The LQG controller response is very smooth and servo tracking of process variable is good. Initially the set point for h1 and h2 are about 24cm and 14cm respectively. At the time of 500s, the height of the tank1 and tank2 are 20cm and 18cm changed respectively, the controller keeps on tracking the changed setpoint, and disturbances applied at 400sec, 800 sec to check the behavior of the controller. The LQG controller tracks the setpoint and rejects the disturbance effectively. The performance of LQG controller is compared with PI controller and tabulated in table.5. The disturbance rejection of LQG is guaranteed.

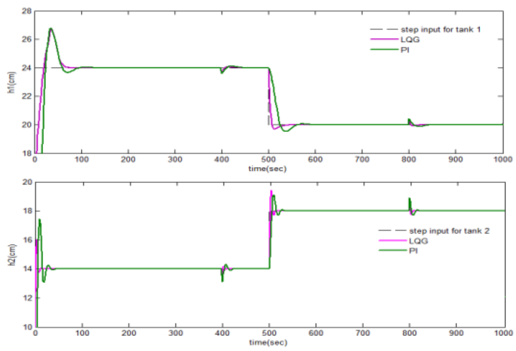

Fig 7 : Servo and Regulatory response of LQG controller

Table.5 Performance Indices for Servo and Regulatory Response of LQG and PI controller

Control Strategy		IAE	ISE	ITAE
PI	Loop1	336.2	3398	28680
	Loop2	110.2	752.9	13030
LQG	Loop1	256	1318	18920
	Loop2	76.49	494.8	9912

7. CONCLUSION

A Linear Quadratic Gaussian (LQG) control design for two interacting conical tank system has been investigated to achieve robust properties. The Linear Quadratic Regulator (LQR) designed by minimizing a performance criterion based on comfort and controller energy considerations; and than a Kalman filter (KF) is used to estimate the state of the system. However, the objective of this paper has been to examine the use of optimal state-feedback controllers in TICTP. The integrator is added for servo tracking, integral constant is based on trial and error method. There is no standard method available for selection of Q and R. Here, based on the impact of states on process output, the Q and R matrix selected. The LQG controller performance is compared with ZN tuned PI controller. This paper has shown the dynamic behaviour of nonlinear multivariable system. The simulation results for variety of input processes shows that the method adopted in this research work gives better controller performance.

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