Newton's-Like Method for Solving Systems of Nonlinear Equations with Singular Jacobian

H. A. Aisha
Department of Mathematics, Faculty of Science, Bayero University Kano, Nigeria

W. L. Fatima
Department of Mathematics, Faculty of Science, Bayero University Kano, Nigeria

M. Y. Waziri
Department of Mathematics, Faculty of Science, Bayero University Kano, Nigeria

ABSTRACT
It is well known that when the Jacobian of nonlinear systems is nonsingular in the neighborhood of the solution, the convergence of Newton method is guaranteed and the rate is quadratic.

Violating this condition, i.e. the Jacobian to be singular the convergence may be unsatisfactory and may even be lost. In this paper we present a modification of Newton's method via extra updating for nonlinear equations with singular Jacobian which is very much faster and significantly cheaper than classical Newton method. Numerical experiments are carried out which shows that, the proposed method is very encouraging

Keywords
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1. INTRODUCTION
Many real-life problems (e.g Robotics, Radioactive transfer, Chemistry, Economics, operational research, physics, statistics, engineering, and social sciences e.t.c) require the solution of systems of nonlinear equations. Consider the problem

$$F(x) = 0,$$
(1)

Where $F : R^n \to R^n$ s a nonlinear mapping. Often, the mapping, F is assumed to satisfying the following assumptions:

A1. F is continuously differentiable in a open neighborhood S of a solution $x^* \in S$ of the system (1.1),

A2. There exists a solution x^* where $F(x^*) = 0$

A3. $F'(x^*) \neq 0$ is invertible.

The best approach for finding the solution to (1), is the Newton's method. The method produces an iterative sequence $\{x_k\}$ from any given initial guess x0 in the neighborhood of solution, through the following stages:

Algorithm 1 (Newton's Method)

Given x_0

solve s_k for k = 0, 1, 2, ...

$$F'(x_k)s_k = -F(x_k)$$

Update

$$x_{k+1} = x_k + s_k$$

where s_k is the Newton correction. Among its attractive feature is that the convergence rate is quadratic from any initial point x_0 in the neighborhood of solution whenever the Jacobian matrix $F'(x^*)$ is nonsingular at a solution [10, 7, 8],

$$\left\|x_{k+1} - x^*\right\| \leq h\left\|x_k - x^*\right\|^2 \qquad (2)$$

For some h. Nevertheless, Newton's method requires the computation of the matrix which entails the first-order derivatives of the systems. In practice, computations of some functions derivatives are quite costly and sometime they are not available or could not be done precisely. In this case Newton's method cannot be applied directly [9, 12, 3].To overcome such difficulty the simple modification on the Newton's method is the Chord Newton method. This method for the determination of solution x^* is given by:

Algorithm 2 (Fixed Newton's Method)

Given x_0

solve s_k for k = 0, 1, 2, ...

$$F'(x_0)s_k = -F(x_k)$$

Update

$$x_{k+1} = x_k + s_k$$

The method avoids computation and storing the Jacobian in each iterations (except at $k = 0$). However it still consumes more CPU time as the system's dimension increases. Quasi-Newton method is another variant of Newton-type methods, it replaces the Jacobian or its inverse with an approximation which can be updated at each iteration [11, 1, 2] and its updating scheme is given by:

Algorithm 3 (Quasi Newton's Method)

Given x_0

Solve s_k for k = 0, 1, 2, ...,

$$B(x_k)s_k = - F(x_k)$$

Update

$$x_{k+1} = x_k + s_k$$

Where the matrix B_k is the approximation of the Jacobian at x_k.

Moreover, must of the existing methods are based on Newton's approach provided the Jacobian is nonsingular. It is well known that, the weakness of Newton's method arise from the non- singularity of the Jacobian matrix in the neighborhood of the solution for successful quadratic rate of convergence [10, 6]. Disobeying this condition, i.e. the Jacobian to be singular the convergence is unsatisfactory and may even be lost. Based on this fact, Newton's-type method that required Jacobian computation may not be a good candidate for solving singular nonlinear equations [3, 5]. This is what inspires us to suggest an extra updating strategy Newton's method for solving nonlinear equations with singular Jacobian singular. The anticipation has been to bypass the point at which the Jacobian is singular. The method proposed in this work is computationally cheaper than classical Newton's method. The organization of this paper is as follows: in section 2, we present our propose method. Some numerical results are reported in section 3, and Conclusion in section 4.

2. MATERIAL AND METHOD FOR THE PROPOSED SCHEME

A new approach for solving nonlinear equations with singular Jacobian has been presented in this section. We continue in the sprit of Newton's method by letting identity matrix to be the initial Jacobian approximation. Then continue the iterations as New- tonian . The anticipation has been to bypass the points in which the Jacobian is singular.

It is vital to mention that, [5] have reported that, the undesirable performance behaviors of Newton's chord methodespecially when solving high dimensional systems of nonlinear equations is associated with the insufficient Jacobian information in each iteration. The validation associated to our procedure is to enhance the convergence properties as well as improving numerical stability. This is made pos- sible by employing Mid- Point strategy on the iterates . With this scheme, we expect our method to yields a significant reduction in CPU time consumption and number of iteration compared to Fixed Newton's method.

Algorithm 4 (Extra Updating Newton's Scheme(EUNM)**)**

Step 1: Given x_o, $M_0 = I_n \varepsilon$ and set k=0.

Step 2: Compute $F(x_k)$

Step 3: Check stopping condition., i.e $\|F(x_k)\| \leq 10^{-3}$, If yes stop, else go to Step

Step 4: Set $J(x_k) = I_n$ for $k = 0$, else $J(x_k)$ is the true Jacobian for $k > 0$

Step 5: Set $x_{k+1} = x_k - J(x_k)^{-1} F(x_k)$,

Step 6: Set k = k+1 and go back to step 2.

3. COMPUTATIONAL EXPERIMENT

To demonstrate the proposed method and also to illustrate the advantages of EUNM method, four (4) problems were solved. We present the first problem, which cannot be solved by Newton's method due to the restrictions (Singular Jacobian). The methods considered are Newton's Method (NM) and Fixed Newton's method (FN) and EUNM (stands for method proposed in this paper).

The identity matrix has been chosen as an initial approximate Jacobian. Test problems are given as follows.

Problem 1:

$$f(x) = \begin{cases} 2x_1^2 \\ x_1^2 + x_2^2 \end{cases}$$

x_0 = (0,1), (0,3)

Problem 2:

$$f(x) = \begin{cases} e^{x_1} - x_2 - 1 \\ x_1^2 + x_2^2 \end{cases}$$

x_0 = (0,0.5), (0,1)

Problem 3:

$$f(x) = \begin{cases} x_1^2 \\ 3x_1^2 + 4x_2^2 \end{cases}$$

x_0 = (0.5,0), (0.7,0)

The Jacobian of the benchmark problems are:

Problem 1

$$J(X_n) = \begin{bmatrix} 4x_1 & 0 \\ 2x_1 & 2x_2 \end{bmatrix}$$

Problem 2

$$J(x_n) = \begin{bmatrix} e^{x_1} & -1 \\ 1 & -1 \end{bmatrix}$$

Problem 3

$$J(x_n) = \begin{bmatrix} 2x_1 & 0 \\ 6x_1 & 8x_2 \end{bmatrix}$$

Proble ms	x_0	**NM**	**FNM**	**EUNM**
1	(0,1) (0,3)	- -	- -	1 13
2	(0,0.5) (0,1)	- -	- -	5 6
3	(0.5,0) (0.7,0)	- -	- -	7 8

The symbol "-" indicates failure due to singularity while the acronyms NM is for Newton's method, FNM for Fixed Newton's Method and EUNM for our proposed method. Considering the initial guess x_0, in, the $J(x_0)$ are Singular. Therefore, NM and FN method due to $J(x_0)$ requirement at x_0 cannot handle problems 1-3. To bay pass the point of singularity, we proposed to let $J(x_0)$to be identity matrix at $k=0$. The numerical results of the three (3) methods are reported in Table 1 for number of iterations. We can easily see from the Table that, only the EUNM method attempted to solve the system of nonlinear equations. In particular, the EUNM scheme is superior to Newton and Fixed Newton method for all the tested benchmarks problems with their respective initial guesses.

4. CONCLUSION

A modification of Newton method for solving nonlinear systems with singular Jacobian of the solution is presented (EUNM). It is well known that the convergence of Newton method in solving nonlinear equations with singular Jacobian is unsatisfactory and may even fail. We have presented a new algorithm by approximating the Jacobian at initial point x_0 with identity matrix. The fact that the EUNM method has a low computational cost and low storage requirements associated with building the approximation of the Jacobian at x0 is a clear advantage of our scheme over NM and FN methods. It is worth mentioning that the EUNM method is the only method that solves the problems and still maintainings the good accuracy of the numerical result. Hence, we conclude that,

EUNM is a good alternative to Newton method and Fixed Newton, especially when the Jacobian is singular at a solution.

5. REFERENCES

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