ABSTRACT

In path factorization, Ushio K [1] gave the necessary and sufficient conditions for $P_k$-design when $k$ is odd. $P_{2p}$-factorization of a complete bipartite graph for $p$, an integer was studied by Wang [2]. Further, Beilng [3] extended the work of Wang [2], and studied $P_{2k}$-factorization of complete bipartite multigraphs. For even value of $k$ in $P_k$-factorization the spectrum problem is completely solved [1, 2, 3]. However, for odd value of $k$ i.e. $P_5, P_7, P_9, P_{11}$ and $P_{4k-1}$, the path factorization have been studied by a number of researchers [4, 5, 6, 7, 8]. The necessary and sufficient conditions for the existences of $P_5$-factorization of symmetric complete bipartite digraph were given by Du B [9]. Earlier we have discussed the necessary and sufficient conditions for the existence of $K_{m,n}$-factorization [10, 11]. Now, in the present paper, we give the necessary and sufficient conditions for the existence of $P_{4k-1}$-factorization of symmetric complete bipartite digraph of $K_{m,n}$.

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Key words: Complete bipartite Graph, Factorization of Graph, Symmetric Graph.

1. INTRODUCTION

Let $K_{m,n}$ be a complete bipartiite symmetric digraph with two partite sets having $m$ and $n$ vertices. A spanning subgraph $\vec{F}$ of $K_{m,n}$ is called a path factor if each component of $\vec{F}$ is a path of order at least two. In particular, a spanning subgraph $\vec{F}$ of $K_{m,n}$ is called a $\vec{F}_{4k-1}$-factor if each component of $\vec{F}$ is isomorphic to $\vec{P}_{4k-1}$. If $K_{m,n}$ is expressed as an arc disjoint sum of $\vec{P}_{4k-1}$-factors, then this sum is called $\vec{P}_{4k-1}$-factorization of $K_{m,n}$. Here, we take path of order $4k-1$. A $\vec{P}_{4k-1}$ is the directed path on $4k-1$ vertices.

2. MATHEMATICAL ANALYSIS

The necessary and sufficient conditions for the existence of $\vec{P}_{4k-1}$-factorization of complete bipartite symmetric digraph are given below in theorem 2.1.

Theorem 2.1: Let $m$ and $n$ be the positive integers then $K_{m,n}$ has a $\vec{P}_{4k-1}$-factorization if:

(1) $2kn \geq (2k-1)m$,
(2) $2kn \geq (2k-1)n$,

(3) $m + n \equiv 0 \pmod{4k-1}$, and
(4) $\frac{(4k-1)mn}{(2k-1)(m+n)}$ is an integer.

Proof of necessity of theorem 2.1

Proof: Let $r$ be the number of $\vec{P}_{4k-1}$-factor in the factorization, and $e$ be the number of copies of $\vec{P}_{4k-1}$-factor in a factorization, which can be computed by using

\[ r = \frac{(4k-1)mn}{(2k-1)(m+n)} \quad \text{...(1)} \]

and

\[ e = \frac{m+n}{4k-1} \quad \text{...(2)} \]

respectively.

Obviously, $r$ and $e$ will be integers. Thus conditions (3) and (4) in theorem 2.1 are necessary. Let $a$ and $b$ be the number of copies of $\vec{P}_{4k-1}$ with its end points in $Y$ and $X$, respectively in a particular $\vec{P}_{4k-1}$-factor. Then by simple arithmetic we can obtain, $2kb + (2k-1)a = m$ and $2ka + (2k-1)b = n$.

From this, we can compute $a$ and $b$ which are as follows:

\[ a = \frac{2kn - (2k-1)m}{4k-1} \quad \text{...(3)} \]

\[ b = \frac{2km - (2k-1)n}{4k-1} \quad \text{...(4)} \]

Since, by definition $a$ and $b$ are integers, therefore equation (1) and (2) imply,

\[ \frac{2kn - (2k-1)m}{4k-1} \geq 0 \]

and

\[ \frac{2km - (2k-1)n}{4k-1} \geq 0, \]

this implies that $2kn \geq (2k-1)m$ that $2kn \geq (2k-1)n$. Therefore condition (1) and
(2) in theorem 2.1 are necessary. This proves the necessity of theorem 2.1.

**Proof of sufficiency of theorem 2.1**

Further, we need the following number theoretic result (lemma 2.2) to prove the sufficiency of theorem 2.1. Its proof can be found in any good text related to number theory [12].

**Lemma 2.2:** If \( \gcd(xv, uy) = 1 \) then \( \gcd(uv, ux + vy) = 1 \), where \( x, y, u \) and \( v \) are positive integers.

We prove the following result of lemma 2.3, which will be used further.

**Lemma 2.3:** If \( K_{m,n} \) has \( P_{4k-1} \) factorization, then \( K_{m,n} \) has \( P_{4k-1} \) factorization for every positive integer \( s \).

**Proof:** Let \( K_{m,n} \) be a \(-1\)-factorable [13] and \( \{F_1, F_2, ..., F_s\} \) be a \(-1\)-factorization of \( K_{m,n} \). For each \( i \) with \( 1 \leq i \leq s \), replace every edge of \( F_i \) by a \(-1\)-factorable subgraph \( G_i \) of \( K_{m,n} \) such that the graph \( G_i \)'s \( (1 \leq i \leq s) \) are pair wise edge disjoint, and there union is \( K_{m,n} \). Hence \( K_{m,n} \) has \( P_{4k-1} \) factorization, it is clear that \( G_i \) is also \( P_{4k-1} \)-factorable, and hence \( K_{m,n} \) has \( P_{4k-1} \) factorization. Now to prove the sufficiency of theorem 2.1, there are three cases to consider:

**Case (i) \( 2kn = (2k-1)n \):** In this case from lemma 2, \( K_{m,n} \) has \( P_{4k-1} \) factorization. Consider the trivial case at \( k = 1, m = 1 \) and \( n = 2 \), then number of copy \( e = 1 \) and total number of factor \( r = 2 \). Path factor is given below:

![Diagram](image)

**Case (ii) \( 2km = (2k-1)m \):** Obviously, \( K_{m,n} \) has \( P_{4k-1} \) factorization since in this case position of \( m \) and \( n \) changes only from previous case.

**Case (iii): \( 2km > (2k-1)n \) and \( 2kn > (2k-1)m \):** In this case, let 

\[
a = \frac{2kn - (2k-1)m}{4k-1}, \quad b = \frac{2km - (2k-1)n}{4k-1}.
\]

Case (i): \( 2kn = (2k-1)m \); In this case, \( a \) is a positive integer and \( a \) and \( b \) are integers. Then from condition (1)-(4) of theorem 2.1, \( a, b, e \) and \( r \) are integers and \( 0 < m < n \) and \( 0 < b < n \). As obtained previously \( 2kb + (2k-1)a = m \) and \( 2ka + (2k-1)b = n \). Hence

\[
r = \frac{4k-1}{4k-1}(m+n), \quad e = \frac{m+n}{4k-1}.
\]

Then from condition (1)-(4) of theorem 2.1, \( a, b, e \) and \( r \) are integers and \( 0 < m < n \) and \( 0 < b < n \). As obtained previously \( 2kb + (2k-1)a = m \) and \( 2ka + (2k-1)b = n \). Hence

\[
r = 2ka + (2k-1)b + \frac{ab}{2k-1}.
\]

Let

\[
z = \frac{ab}{2k-1}.
\]

These equality implies the following equality:

\[
d = \frac{(2k-1)2kp + (2k-1)qz}{pq},
\]

\[
r = \frac{(p+q)4k^2p + (2k-1)^2qz}{pq},
\]

\[
m = \frac{(2k-1)(p+q)2kp + (2k-1)qz}{pq},
\]

\[
n = \frac{4k^2p + (2k-1)^2qz}{2kpq}.
\]

These equality implies the following equality:

\[
a = \frac{p2kp + (2k-1)qz}{pq},
\]

\[
b = \frac{(2k-1)q2kp + (2k-1)qz}{2kpq}.
\]

Let \( k = \frac{1}{p_1p_2...p_{k}b}, \frac{1}{p_1p_2...p_{k}b}, ..., \frac{1}{p_1p_2...p_{k}b} \) where \( p_1, p_2, ..., p_k \) are distinct prime number.

Let \( k_1, k_2, ..., k_r \) are positive number, and \( 2k = k_1^1, k_2^1, ..., k_r^1 \) are positive integers, where \( h_1, h_2, ..., h_w \) are positive integer.

If

\[
\gcd(p, 2k-1) = p_{1}^{h_1}p_{2}^{h_2}...p_{k}^{h_k} \cdot p_{k+1}^{2k-1}\cdot p_{k+2}\cdot p_{k+3}\cdot ...\cdot p_{2k-2},
\]

Where \( 1 \leq \alpha \leq \beta \leq 

\[
0 \leq n_k \leq \gamma \text{when } 0 \leq j \leq \alpha \) or

\[
0 \leq i_j \leq \kappa(j) \text{when } \alpha + 1 \leq j \leq \beta \),

\[
gecd(q, 4k^2) = q_{1}^{h_1}q_{2}^{h_2}...q_{k}^{h_k} \cdot q_{k+1}^{2k-1}\cdot q_{k+2}\cdot q_{k+3}\cdot ...\cdot q_{2k-2},
\]

Where \( 1 \leq \mu \leq \theta \leq \nu \),

\[
0 \leq j_i \leq h_i \text{when } 0 \leq i \leq \mu \) or

\[
0 \leq j_i \leq h_i \text{when } \mu + 1 \leq i \leq \nu \),

\[
s = p_{1}^{h_1}p_{2}^{h_2}...p_{k}^{h_k}, \quad t = p_{1}^{h_1}p_{2}^{h_2}...p_{k}^{h_k}, \quad \gamma = p_{k+1}^{h_1}p_{k+2}^{h_2}...p_{2k-2}^{h_k},
\]

\[
u = p_{k+1}^{h_1}p_{k+2}^{h_2}...p_{2k-2}^{h_k}p_{2k-1}^{h_k}.
\]
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\[ w = p_{b+1}^{h_1} p_{b+2}^{h_2} \ldots p_r^{2k_r}, \]

and

\[ s' = a_1 q_1 \ldots q_{\mu-1} q_{\mu} t', \]

\[ u' = q_{\mu+1} q_{\mu+2} \ldots q_r \]

\[ w' = q_{\mu+1}^{h_{\mu+1}} q_{\mu+2}^{h_{\mu+2}} \ldots q_r^{h_r}. \]

Then \( 2k - 1 = stuvw \) and \( 2k = s't'u'v'. \)

Also let \( p = s'uv'^2w^2 \) and \( q = s'u'v'^2w^2q'. \) Now by using \( p, q, (2k - 1) \) and \( 2k \) the parameter \( m, n, a, b \) and \( r \) satisfying the condition \((1) - (4)\) are expressed in the following lemma (2.4). The purpose of lemma (2.4) is to discuss the detail of \( F_{b-1} \) factorization, and reduce it to number of base cases, which are then solve in later lemma.

**Lemma 2.4:**

**Case 1:** If \( t' \equiv 1 \pmod{2} \) and \( v'w' \equiv 1 \pmod{2} \), then

\[ m = stut'(sw^2v'p' + s'u'v'^2w^2q')(twvp + tvwq)wz', \]

\[ n = suvw'w'(st'u'p' + st^2uq')(twv'p + tvwq)wz', \]

\[ a = suvw't'(vw't'p' + tvw'q')wz', \]

\[ b = stuw'w'q'(vw't'p' + tvw'tq')wz'/2, \]

\[ r = t'w'(sw^2v'p' + s'u'v'^2w^2q')(st'u'p' + st^2uq')wz'/2, \]

\[ d = stut'(vw't'p' + tvw'tq')wz'/2, \]

for some positive integer \( z' \).

**Case 2:** If \( t' \equiv 0 \pmod{2} \) and \( v'w' \equiv 1 \pmod{2} \), then

\[ m = stut'(sw^2v'p' + s'u'v'^2w^2q')(twv'p' + tvw'q')wz'/2, \]

\[ n = suvw'w'(st'u'p' + st^2uq')(twv'p' + tvw'q')wz'/2, \]

\[ a = suvw't'(vw't'p' + tvw'q')wz'/2, \]

\[ b = stuw'w'q'(vw't'p' + tvw'tq')wz'/2, \]

\[ r = t'w'(sw^2v'p' + s'u'v'^2w^2q')(st'u'p' + st^2uq')wz'/2, \]

\[ d = stut'(vw't'p' + tvw'tq')wz'/2, \]

for some positive integer \( z' \).

**Case 3:** If \( t' \equiv 1 \pmod{2} \) and \( v'w' \equiv 0 \pmod{2} \), then

\[ m = stut'(sw^2v'p' + s'u'v'^2w^2q')(tvw'p' + tvw'q')wz'/2, \]

\[ n = suvw'w'(st'u'p' + st^2uq')(tvw'p' + tvw'q')wz'/2, \]

\[ a = suvw't'(vw't'p' + tvw'q')wz'/2, \]

\[ b = stuw'w'q'(vw't'p' + tvw'tq')wz'/2, \]

\[ r = t'w'(sw^2v'p' + s'u'v'^2w^2q')(st'u'p' + st^2uq')wz'/2, \]

\[ d = stut'(vw't'p' + tvw'tq')wz'/2, \]

for some positive integer \( z' \).

**Proof:** Let us assume that \( \gcd(p, (2k - 1)^2) = s'uv'^2w^2 \) and \( \gcd(q, 4k^2) = s'u'v'^2w^2. \)

If \( \gcd(p, q) = 1 \) and \( p = s'uv'^2w^2p' \) and \( q = s'u'v'^2w^2q' \) hold.

then

\[ \gcd(s'uv'^2w^2p', s'u'v'^2w^2q') = 1. \]

Which implies that, if \( \gcd(4k^2p, (2k - 1)^2) = 1 \) and \( (2k - 1) = stuvw \) and

\( 2k = s't'u'v'w' \) hold, then \( \gcd(s't'u'v'p', st^2uq) = 1. \)

Since

\[ r = (sw^2v^2p' + s'u'v'^2w^2q')(st'u'p' + st^2uq)z/pq, \]

is an integer, therefore by using lemma 2.2, we see that \( \gcd(s'uv'^2w^2p', s'u'v'^2w^2q') = 1 \)

implies that

\[ \gcd(s't'u'v'p', st^2uq') = 1 \]

and

\[ \gcd(s't'u'v'p', st^2uq) = 1. \]

Since \( r \) is an integer therefore \( \frac{r}{pq} \) must be an integer.

Let \( z_1 = \frac{r}{pq} \) then we have

\[ d = \frac{stuvw't'p' + tvw'tq'}{tvw'q'}, \]

is an integer.

Now for the values of \( t' \) and \( v'w' \) there are three cases will possible.

**Case 1:** When \( t' \equiv 1 \pmod{2} \) and \( v'w' \equiv 1 \pmod{2} \).

Since,

\[ \gcd(2, v'w') = \gcd(stuv, v'w') = \gcd(vw't'p' + tvw'q', v'w') = 1, \]

therefore \( z_2 = \frac{z_1}{v'w'} \) is an integer. Letting \( z_2 = \frac{z_1}{v'w'} \) we have

\[ b = \frac{stuvw'q'(vw't'p' + tvw'q')z_2}{t}, \]

Since

\[ \gcd(2, t') = \gcd(stuvw'q', t') = \gcd(vw't'p' + tvw'q', t') = 1. \]

Therefore \( z_1 = \frac{z_2}{t'} \) is an integer. Let \( z_1 = \frac{z_2}{t'} \). Then all the values \( m, n, a, b, r \) and \( d \) in case (1) hold.

**Case 2:** When \( t' \equiv 0 \pmod{2} \) and \( v'w' \equiv 1 \pmod{2} \).

Since \( \gcd(2, v'w') = 2, \)

\[ \gcd(stuv, v'w') = \gcd(vw't'p' + tvw'q', v'w') = 1, \]

therefore \( \frac{z_1}{v'w'} \) is an integer. Let \( z_2 = \frac{z_1}{v'w'} \), then

\[ b = \frac{stuvw'q'(vw't'p' + tvw'q')z_2}{2t}. \]
Since \( \gcd(2, t) = 2 \), \( \gcd(stu'w'q', t) = \gcd(vwt'p + tw'q', t) = 1 \), therefore \( \frac{za}{t} \) is an integer. Let \( z = \frac{za}{t} \). Then all the values of \( m, n, a, b, r \) and \( d \) in case (2), hold.

**Case 3:** When \( t \equiv 1 \pmod{2} \), and \( v'w' \equiv 0 \pmod{2} \).

Since 
\[
\gcd(2, v'w') = 2, \quad \gcd(stu', v'w') = \gcd(vwt'p + tw'q', v'w') = 1,
\]
Therefore \( \frac{za}{2t} \) is an integer. Let \( z = \frac{za}{2t} \), then 
\[
b = \frac{stu'w'q'(vwt'p + tw'q')}{2t}.
\]
Since \( \gcd(stu'w'q', t') = \gcd(vwt'p' + tw'q', t') = 1 \), therefore \( \frac{za}{2t} \) is an integer. Let \( z = \frac{za}{2t} \). Then all the values of \( m, n, a, b, r \) and \( d \) in case (3), hold.

This proves the lemma 2.4.

For the parameters \( m \) and \( n \) in lemma 2.4 case (1) - case (3) when \( s = 1 \), we can construct a \( \tilde{P}_{4k-1} \) factorization of \( K_{m,n}^* \).

It is easy to see that the existence of a \( \tilde{P}_{4k-1} \) factorization of \( K_{m,n} \) implies the existence of a \( \tilde{P}_{4k-1} \) factorization of \( K_{m,w} \).

For our main result we need to prove the following lemma:

**Lemma 2.5:** For any positive integers \( s, t, u, v, w, s', t', u', v', w', p \), and \( q \), let 
\[
m = stu(tsw'tq + s'u'w'q')(twp'vq + tw'q'),
\]
\[
n = swtw'q('st^2u'p + s't'uqw'),
\] 
Then \( K_{m,n}^* \) has a \( \tilde{P}_{4k-1} \) factorization if \( s, t, u, v, w + 1 = s't'u'w' \), where \( k - 1 = s = t = u = v + 1 \).

**Proof.** The proof is by construction (case 1 of lemma 2.4). Let 
\[
a = swtwp(vwtp + tw'q),
b = stuwq(vwtp + tw'q'),
\]
\[
r = tsww'(vsw'p + s'u'w'q')(st^2u'p + s't'uqw'),
\]
\[
r_1 = t(s't^2u'p + s't'uqw'),
\]
\[
r_2 = tw(s't^2u'p + s't'uqw').
\]

Let \( X \) and \( Y \) be the two partite sets of vertices of \( K_{m,n}^* \) such that:
\[
X = \{x_{ij} \mid 1 \leq i \leq r_1, 1 \leq j \leq m_0 \},
\]
\[
Y = \{y_{ij} \mid 1 \leq i \leq r_2, 1 \leq j \leq n_0 \}.
\]
Where first subscript of \( x_{ij} \)'s and \( y_{ij} \)'s taken additional modulo \( r_1 \) and \( r_2 \) respectively and the second subscript of \( x_{ij} \)'s and \( y_{ij} \)'s taken additional modulo \( m_0 \) and \( n_0 \) respectively, where \( m_0 = \frac{m}{r_1} \) and \( n_0 = \frac{n}{r_2} \) i.e.,
\[
m_0 = stu(twp'vq + tw'q'),
\]
\[
n_0 = swtwq(twp'vq + tw'q').
\]

Now we construct a model of \( \tilde{P}_{4k-1} \) factor of \( K_{m,n}^* \), here type \( M \) copies of \( \tilde{P}_{4k-1} \) denote the \( \tilde{P}_{4k-1} \) with its end point in \( Y \) and type \( W \) with its end point in \( X \).

Type \( M \) copies of \( \tilde{P}_{4k-1} \).

For each \( i, x, y, z \) and \( x' \), \( 1 \leq i \leq t'p, 1 \leq x \leq t \), let 
\[
f(i, x, y) = swtw^2(x - 1) + suw(x - 1) + y,
\]
\[
g(i, y, z, x) = s'tu'v'w'q'(x - 1) + suw(x - 1) + y + x',
\]
\[
and h(i, x, y, x') = suw(i - 1) + suw(twp' + tw'q')q(x - 1) + y + x' - 1.
\]

Hence set
\[
E = \{ (i, x, y, z) : f(i, x, y), g(i, y, z, x), h(i, x, y, x') \}.
\]

Each of \( \tilde{E}_1 \) consists of \( n_0 \) vertex disjoint type \( M \) copies. And \( \cup_1 \tilde{E}_i \) contains \( a = swtwp(vwtw + tw'q) \) vertex disjoint type \( M \) copies of \( \tilde{P}_{4k-1} \).

Type \( W \) copies \( \tilde{P}_{4k-1} \).

For each \( i, x, y, z \) and \( x' \), \( 1 \leq i \leq v'w'q, 1 \leq x \leq stu \),
\[
1 \leq y \leq swtw, 1 \leq z \leq t, 0 \leq x' \leq 1,
\]
\[
let \psi(i, x, z) = s'tu'v'w'p + st^2u(x - 1) + stu(x - 1) + x,
\]
\[
\varphi(i, x, y, x') = swtw^2tp + s'u'w't'(i - 1) + svw(x - 1) + y + x',
\]
\[
and \Phi(i, x, y, x') = swtw^2tp + x + stu(i - 1) + suvwtwp + tw'q')(y - 1) + x' - 1.
\]

Hence set
\[
E = \{ (i, x, y, z) : f(i, x, y), g(i, y, z, x), h(i, x, y, x') \}.
\]

Each of \( \tilde{E}_i \) consists of \( n_0 \) vertex disjoint type \( W \) copies. And \( \cup_1 \tilde{E}_i \) contains \( a = swtwp(vwtw + tw'q) \) vertex disjoint type \( W \) copies of \( \tilde{P}_{4k-1} \).
It is shown that the graphs $F_{i,j}$ $(1 \leq i \leq r_1, 1 \leq j \leq r_2)$ are edge disjoints factor of $K_{m,n}$ and their union is $K_{m,n}$. Thus $(F_{i,j}; 1 \leq i \leq r_1, 1 \leq j \leq r_2)$ is a $P_{4k-1}$ factorization of $K_{m,n}$. This proves the lemma 2.5.

By similar manner we can also prove the other two cases of lemma 2.4.

Applying lemma 2.3 – 2.4 and 2.5, we see that for parameter $m$ and $n$ satisfying conditions in theorem 2.1, $K_{m,n}$ has a $P_{4k-1}$ factorization.

3. REFERENCES


