Particular Type of Hamiltonian Graphs and their Properties

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ABSTRACT
In this paper, various properties of particular type of Hamiltonian graph and it's edge-disjoint Hamiltonian circuits have been discussed. It has been found that the intersection graph obtained from Euler Diagram is not Hamiltonian. The graph H(3m + 7, 6m + 14) for m ≥ 1, which is planner, regular of degree four, non-bipartite but Hamiltonian graph , has perfect matching 4 with non- repeated edge for simultaneous changes of m= 2n+1 for n≥0.

General Terms
Properties of Hamiltonian graph.

Keywords
Hamiltonian, Regular, Edge-disjoint Hamiltonian circuits, Perfect matching, Intersection graph.

1. INTRODUCTION
Hamiltonian graph plays a very important role in real life's problem. It is known that a Hamiltonian graph is a graph having at least one Hamiltonian circuit. It is a well known classical NP-complete problem theoretically and computationally [1- 4]. Extensive research on NP-Complete problem has been done on this topic. Various works related to find the number of Hamiltonian circuit of a Hamiltonian graph have been found. Besides, finding the edge-disjoint Hamiltonian circuits of a Hamiltonian graph is an important unsolved problem. The importance of Hamiltonian graphs has been found in case of traveling salesman problem if the graph is weighted graph.

Various properties of Hamiltonian graph obtained from various sub-graph of the complete graph K_{2m+3} for m ≥ 2 have been discussed by Jayanta. et al [5]. Besides, the structure of non-isomorphic Hamiltonian sub-graph of the form H(2m + 3, 6m + 3) for m ≥ 2 has been obtained by Kalita [6]. It has been found that the non-isomorphic Hamiltonian sub-graph of the complete graph K_{2m+3} for m ≥ 2, plays a role in getting the various structure of metal atom cluster compound in chemistry.

The regular planar – sub-graph of the complete graph K_{2m+2} for m ≥ 2 has also been discussed and their application has been considered to find the solution of TSP by Anupam Dutta et al [7]. Some theoretical properties for regular graphs have also been discussed.

Many results have been coming since the existence of the Hamiltonian graph. Shih et al[8] solved the Hamiltonian cycle problem on circular-arc graphs in $O(n^2 \log n)$ time , where n is the number of vertices of the input graph. Ruo-Wei Hung, Maw-Shang Changs, and Chi-HyiLaio [9] solved the same problem on circular-arc graphs in O(Δn) time, where Δ denotes the maximum degree of the input graph. Hamiltonicity of regular graphs was studied by Rui Wang,

Francis C. M. Lau, Yingchao Zhao [10] and many theoretical results have also been developed. But, it has been found that the naive algorithm for finding Hamiltonian path does not run in polynomial time [11]. A polynomial time algorithm for Hamilton cycle (path) was developed by Lizhi Du [12]. Lazhi Du mentioned there that the Hamiltonian circuit problem is still NP-Complete in case of the vertices having degree 3 or 4 or 5.Recently, Ruo-Wei. Hung. etal [26] discussed the edge disjoint Hamiltonian cycles related with the Augmented Cubes. In addition to this he found that two equal path partition in Augmented Cubes exist..Till now, there is no efficient algorithm for Hamiltonian path / cycle, even though many attempts have been made.

It is known that two Hamiltonian cycles are called edge disjoint if there is no common edge in those two Hamiltonian cycles. There are many applications of edge-disjoined Hamiltonian cycles, quite a few of them are in parallel computing and in interconnected network [13-14]. Two edge-disjoined Hamiltonian cycle problem is also NP-complete [15].

The number of edge-disjoined Hamiltonian circuits [16] in the complete graph K_n of odd order is found as (n-1)/2. S. K. Ayyaswamy et al discussed a method of finding the number of edge-disjoined Hamiltonian circuits for complete graph of even order [17]. Anna Gorbenko et al discussed the problem of finding two edge-disjoined Hamiltonian cycles [18] considering the logical models of the graph. It has also been found that any graph of the form H(2m+3,6m+6) for m≥2 having minimum degree 4 and maximum degree 2m+2 for m ≥2 has two edge disjoint Hamiltonian cycles [19] .Variation of path partition problems of a graph has been investigated by R.W .Hung et al [20] for the circular arc graphs . Besides, many authors have found many results related to path partition [21 - 25].

The paper is organized into three sections. In section 1, the introduction including previous work has been discussed. In section 2, some theories related to Hamiltonian graph are developed. In addition to this, in section 3, the two–equal path partitions and perfect matching have been focused.

2. THEOREMS RELATED TO HAMILTONIAN GRAPH
The following theorems are considered for different types of Hamiltonian graphs.

2.1 Theorem
The graph G (2n + 2, 3n + 3) for n ≥ 1, which is regular of degree three, non-bipartite and planner, is always Hamiltonian.

Proof: When we consider for n=1, then it is cleared that the graph G(2n + 2, 3n + 3) for n ≥ 1, has at least one Hamiltonian circuit. [Figure1].
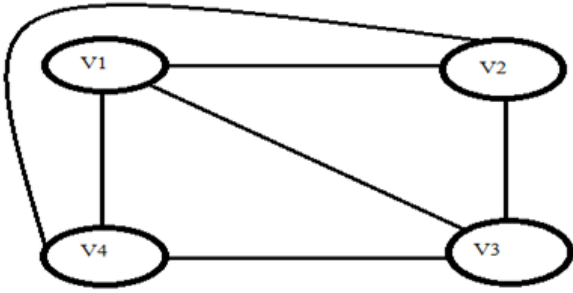

Figure 1

Again if we consider the value of n=2, then we can construct a graph of six vertices and 9 edges, which is also a regular graph of degree 3 and this graph also contains at least one Hamiltonian circuit and hence it is Hamiltonian. Continuing the process for different values of n ≥3, keeping in mind that the resulted graph as regular of degree three , it can be shown that the graph is Hamiltonian.

2.2 Theorem

The regular graph G (4n + 4, 6n + 6) for n ≥ 1, of degree three and planar of odd number of regions having four edges when n = 1 and only two regions covered by 2m + 4 edges for m ≥ 1 for simultaneous changes of n ≥ 2 is always bi-colorable.

Proof: Let us considered the following graph having odd numbers of regions greater than 3, covered by four edges as shown in figure-2 for n = 1.

Let us now color the vertices of the graph. The vertex V_0 is colored by the color C1 and the vertex V_1 is colored by the color C2. It can be shown that only two colors (C1 & C2) are necessary to color the graph of figure 2 properly according to the coloring property as shown in figure-2.

Now let us considering the values of n = 2 for simultaneous changes of m = 1, the graph of figure-3 can be constructed which contains five regions covering of four edges and two regions covering of six edges. Now this graph can also be colored only by two colors C1 & C2.

Again constructing the graph for n = 3 and m=2, it can be shown that the graph can be colored with two colors (C1 & C2).Now proceeding in the same way of coloring the graph for other values of n and m, we immediately can show that only two colors are necessary for coloring the graph structure, which clears the theorem.
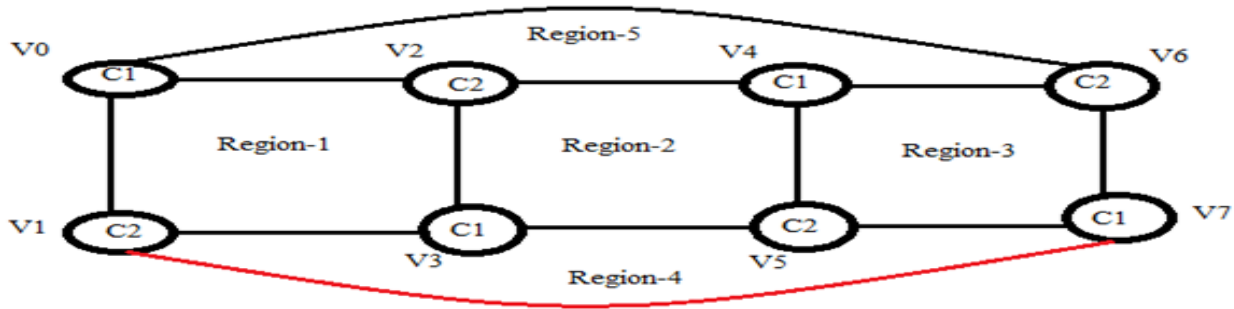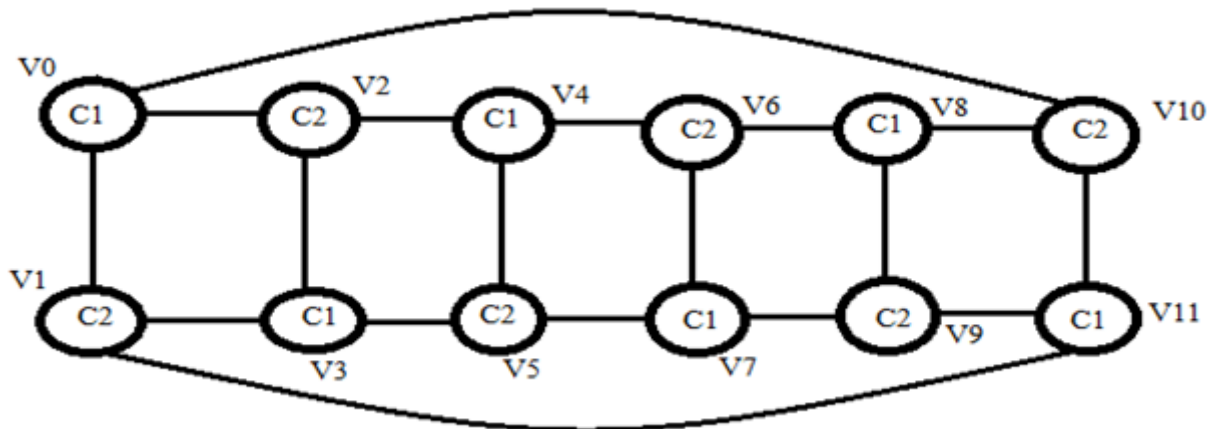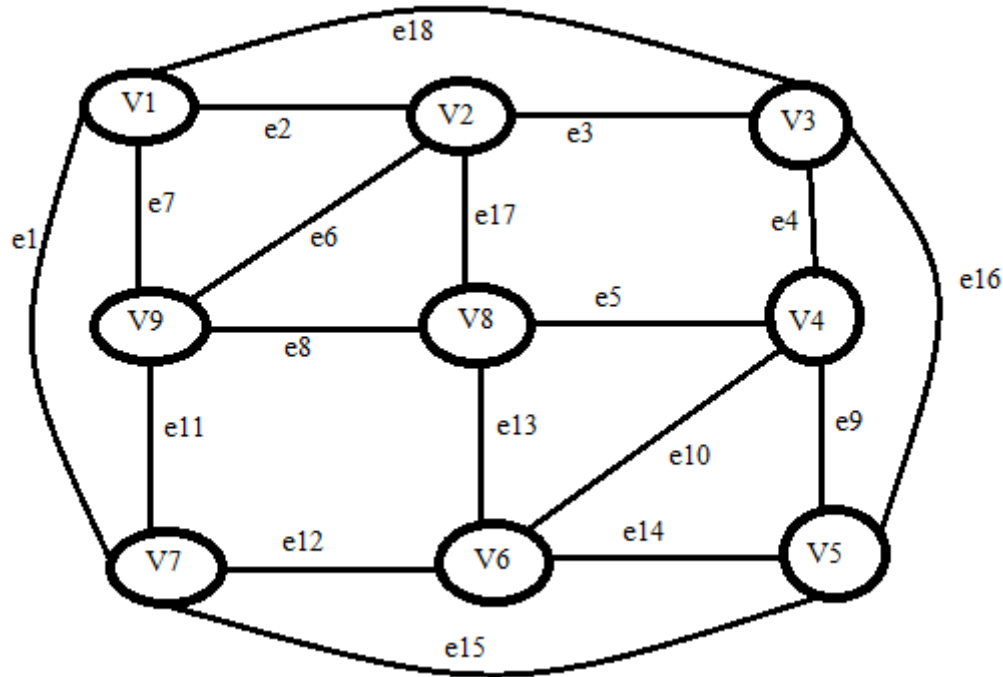

Figure 2

Figure 3

Figure 4

2.3 Theorem

The graph $G(3m + 6, 12 + 6m)$ for $m \geq 1$, which is regular of degree four, non-bipartite and planner, has two edge-disjoint Hamiltonian cycles.

Proof: The graph $G(3m + 6, 12 + 6m)$ is a planner graph [Figure-4 is for m=1]], as there is no intersection between their edges and it is a regular graph of degree four. The graph $G(3m + 6, 12 + 6m)$ for $m \geq 1$can be constructed as regular graph of degree four .This graph has at least one Hamiltonian circuit and hence it is Hamiltonian. Now our aim is to find how many edge-disjoint Hamiltonian circuits in it. Let us consider one Hamiltonian circuit – $V_1e_{18}V_3e_3V_2e_{17}V_8e_{13}V_6e_{10}V_4e_9V_5e_{15}V_7e_{11}V_9e_7V_1$,where V_1, $V_3,V_2,$ $V_8,$ $V_6,$ $V_4,$ $V_5,$ $V_7,$ $V_9$are the vertices of the graph and $e_{18},e_3,e_{17},e_{13},e_{10},e_9,e_{15},e_{11},e_7$ are the edges of the graph.

Let us consider another Hamiltonian circuit $V_1e_2V_2e_6V_9e_8V_8e_5V_4e_4V_3e_{16}V_5e_{14}V_6e_{12}V_7e_1V_1$, where

$V_1,V_2,V_9,V_8,$ V_4,V_3,V_5 ,V_6,V_7 , are the vertices of the graph and $e_2,e_6,$ $e_8,$ $e_5,$ $e_4,$ $e_{16},$ $e_{14},$ e_{12}, are the edges of the graph. It is found that the edges $e_{18},$ $e_3,$ $e_{17,}$ $e_{13},$ $e_{10},$ $e_9,$ $e_{15},$ $e_{11,}$ e_7 formed for the first circuit and the edges $e_2,e_6,$ $e_8,$ $e_5,$ $e_4,$ $e_{16},$ $e_{14},$ $e_{12},$ formed for the second Hamiltonian circuit and these give all the edges $12 + 6m$, visiting all the vertices exactly once. For $m = 1$, the figure 4 clears the statement, which has only two edge-disjoint Hamiltonian circuits.

Constructing the graph for $m \geq 2$, it can be shown in the same way that the graph structure has only two edge-disjoint Hamiltonian circuits, which complete the proof.

2.4 Theorem

The graph $G (3m + 7, 6m + 14)$ for $m \geq 1$, which is planner and regular of degree 4, and non-bipartite, has two edge-disjoint Hamiltonian circuits.

Figure 5

Proof: The graph G(3m + 7, 6m + 14) is a planner graph for m ≥ 1.Since degree of each vertex is 4, hence it is regular of degree 4, and it is non-bipartite as it contains triangle. Hence this graph is regular of degree 4, non-bipartite, planar.

Now we have to show that this graph has two edge-disjoint Hamiltonian circuits. Let us consider first Hamiltonian circuit of the graph of figure 5, as $V_1e_1V_{10}e_{20}V_5e_8V_3e_5V_2e_4V_7e_{16}V_4e_{10}V_6e_{13}V_8e_{14}V_9e_{17}V_1$ where $V_1, V_{10}, V_5, V_3, V_2, V_7, V_4, V_6, V_8, V_9$, are the vertices and $e_1, e_{20}, e_8, e_5, e_4, e_{16}, e_{10}, e_{13}, e_{14}, e_{17}$, are the edges of the graph. Let us consider another Hamiltonian circuit of the graph of figure 5 as $V_1e_2V_2e_3V_9e_{15}V_7e_{11}V_6e_{12}V_5e_9V_4e_7V_3e_6V_{10}e_{18}V_8e_{19}V_1$, where V_1 $V_2 V_9 V_7, V_6, V_5, V_4, V_3, V_{10}, V_8$ are the vertices of the graph and e_2, e_3, e_{15}, e_{11}, e_{12}, e_9, e_7, e_6, e_{18}, e_{19} are the edges of the graph. And edges of the two Hamiltonian circuits give the all the edges 6m + 14.This means that figure 5 clears the statement for m=1.

Now constructing the graph for m ≥2, it can be shown that the graph structure has only two edge- disjoint Hamiltonian circuits, which clears the statement.

We now find another structure of graph adding one vertex to the graph (3m + 6, 12 + 6m) for m≥ 1, which is non-regular and hence the further theorem exist as follows.

2.5 Theorem

If one vertex is added outer side the region of the graph H(3m + 6, 6m + 12) for m ≥ 1, making the degree of added vertex of degree four, the new graph H(3m + 7, 6m + 16) for m ≥ 1, which is planar, non-regular, non-bipartite but always Hamiltonian graph.

Proof: Since there are 3m + 7 vertices and 6m + 16 edges in the graph. And e ≤ 3n – 6 (i.e. 6m + 16 ≤ 3(3m + 7) – 6 => 6m + 16 ≤ 9m + 15). The graph is planar having the vertices of degree are either 4 or 5, hence the graph is non-regular. The graph contains triangle hence the graph is non-bipartite. The graph contains at least one Hamiltonian cycle as shown in the figure 6. This has clarified the statement for m=1.Constructing graph for m ≥ 2, it can be shown in the same way that the graph structure H(3m +7, 6m + 16) is planar, non-regular, non-bipartite but always Hamiltonian graph, which clears the statement.
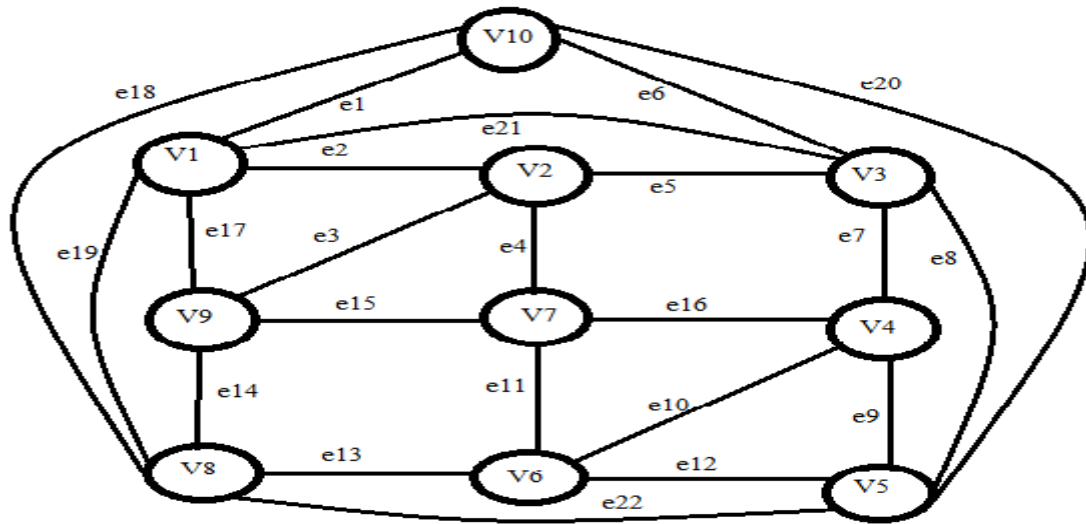

Figure 6

We now find one important graph which is not Hamiltonian, which has been discussed in [26] obtained from particular type of the Euler diagram.

2.6 Theorem

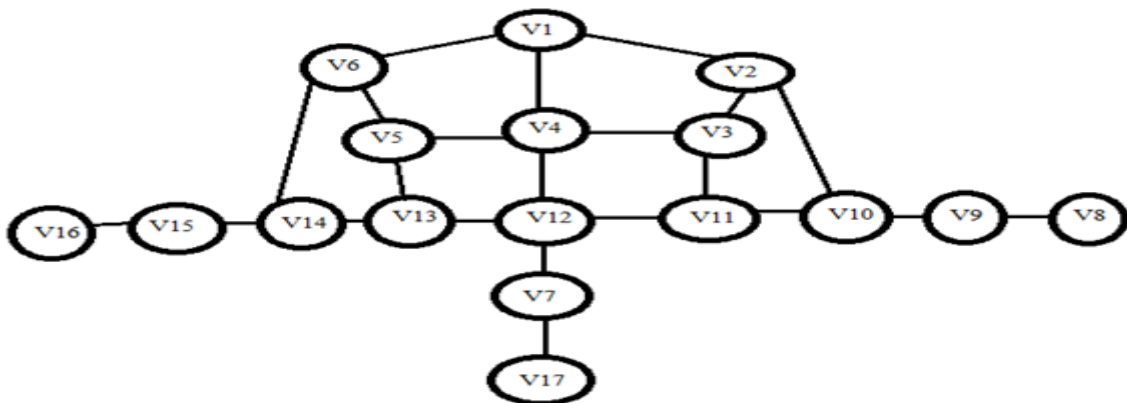Intersection graph obtained from Euler diagram is not Hamiltonian.

Figure 7

Proof: The diagram of figure 7 is considered from[27]. This is an intersection graph obtained from the Euler diagram. There are 17 vertices and the minimum degree $\delta(G)=2$. And $n(G)= 17$, since $\delta(G)=2$ is not greater than equal to $n(G)/2$ [26], hence the intersection graph obtained from the Euler diagram is not Hamiltonian.

3 THEOREMS RELATED TO EQUAL PATH PARTITIONS AND PERFECT MATCHING

Three theorems have been stated and proved for the graphs of the form G $(3m + 6, 12 + 6m)$ and G $(3m + 7, 6m + 14)$ for m ≥ 1, which are regular of degree four, non-bipartite and planner regarding the equal path partitions and for the graph G $(3m + 7, 6m + 14)$ for m ≥ 1, which is planner and regular of degree 4, and non-bipartite, regarding the existence of perfect matching.

3.1   Theorem

The graph structure G $(3m + 6, 12 + 6m)$ for m ≥ 1, which is regular of degree four, non-bipartite and planner, has two-equal path partitions.
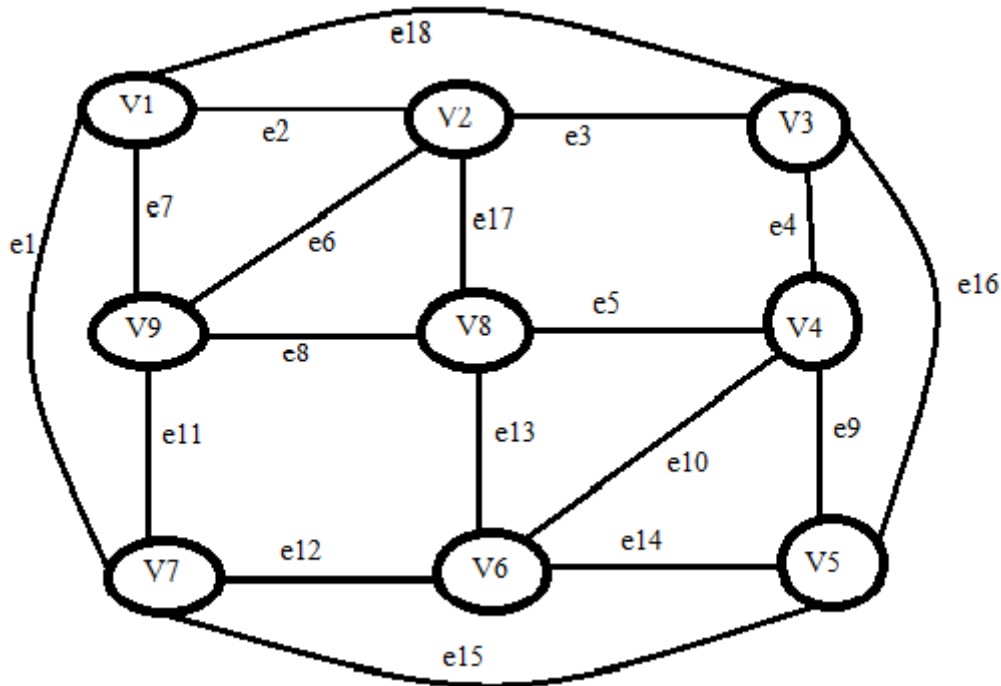
Proof: We consider the graph as shown in the figure 8, G $(3m + 6, 12 + 6m)$ for m $=1$.

For any distinct pair of nodes (v_1, v_2) and (v_4, v_5), there exist two edge-disjoint Hamiltonian paths,

$B = V_1e_1V_7e_{12}V_6e_{14}V_5e_{16}V_3e_4V_4e_5V_8e_8V_9e_6V_2$ and
$R = V_4e_{10}V_6e_{13}V_8e_{17}V_2e_3V_3e_{18}V_1e_7V_9e_{11}V_7e_{15}V_5$ of the graph, satisfying

(1) that is starting the path $B=v_1$, and end of the path $B = v_2$,
(2). Start of the path $R = v_4$, and end of the path $R = v_5$.
(3) $|B|=8 = |R|$
(4) $V(R) \cup V(B) = V (G)$.

This clears the statement for m=1. Let us consider that the result is true for m=k. Therefore the graph $G(3k+6,12+6k)$ has two equal path partitions. Now if we put the vales of m=k+1, the graph structure will be G $(3k + 9, 6k + 18)$. Now m = k + 1 ≥ 1, implies that k ≥ 0, and the graph structure G $(3k + 9, 6k + 18)$ has two-equal path partitions. Hence the result is true for all values of m ≥ 1, which clears the statement.

Figure 8

3.2   Theorem

The graph structure G $(3m + 7, 6m + 14)$ for m ≥ 1, which is regular of degree four, non-bipartite and planner, has two-equal path partitions.

Proof: Same as above.(Figure 5).

3.3   Theorem

The graph H$(3m + 7, 6m + 14)$ for m ≥ 1, which is planner, regular of degree four, non-bipartite but Hamiltonian graph , has perfect matching 4 with non- repeated edge for simultaneous changes of m= 2n+1 for n≥0.

Proof: When n=0,then m=1, and the structure of the graph is H(10, 20) as shown in Figure-5 [theorem-4]. From the figure it is clear that there are 4 set of perfect matching and they are $\{e_1, e_3, e_8, e_{16}, e_{13}\}$, $\{e_{18}, e_{17}, e_4, e_7, e_{12}\}$, $\{e_6, e_2, e_{14}, e_{11}, e_9\}$, and $\{e_{20}, e_5, e_{10}, e_{15}, e_{19}\}$ with non-repeated edge. Therefore the theorem is true for m =1. Suppose the result is true for m = k. So the graph structure H(3k + 7, 6k + 14) has 4 set of perfect matching with non-repeated edge. Now putting m = k + 1,for n=k/2, the structure of the graph will be H(3k + 10, 6k + 20). Now m = k + 1 ≥ 1, as the value of m ≥ 1 and this implies that k ≥ 0. Hence the result is true for all values of m ≥ 1 when m =2n+1 for n≥0.

4 CONCLUSION

Some particular type of Hamiltonian graphs have been constructed and their various properties related to edge disjoint Hamiltonian cycles, vertex coloring, equal path partition and perfect matching have been discussed theoretically. It has been found that the application of edge disjoint Hamiltonian cycles has been prevailed in parallel computing and perfect matching, vertex coloring have been used in scheduling.. Hence these types of discussion definitely will help in these lines. In addition, it has been found that the intersection graph obtained from the Euler diagram is not Hamiltonian.

5. ACKNOWLEDGMENTS

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