A Study on Super Vertex Graceful Graphs

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ABSTRACT
In this paper a brief discussion is made on the super vertex graceful graphs. In particular the order and size plays vital role in labelling the graphs. Also an analysis is made on the order of the complete bipartite graphs under super vertex graceful map.

AMS Classification 05C78

Keywords
Complete graphs, Cycles, Complete bipartite graphs, Graceful graphs, Super vertex graceful graphs.

1. INTRODUCTION
A graph labelling serve as useful models for a broad range of applications such as coding theory, X-ray, crystallography, etc. Graceful

labelling was introduced by Rosa in 1967.Since then many types of labelling namely Harmonious labelling, total magic labelling, edge graceful labelling, odd edge graceful labelling, etc were emerged as given in Dynamic survey [7]. Sin Min Lee has introduced super vertex graceful labelling [15]. N.Murugesan, R.Uma [10] have analysed the super vertex gracefulness of complete bipartite graphs. Also they have discussed the amalgamation of graphs under graceful mapping and Fibonacci graceful labelling[9]. Also, N.Murugesan, R.Uma[13] have analysed the role of order and size of cycles, crowns, wheels under super vertex graceful mapping. In this paper an analysis is made on the order size and degree of the graphs under super vertex graceful map.

2. DEFINITION
2.1 Graceful labelling
Let G = (V, E) be a simple graph with p vertices and q edges. Let f:V(G) → {0, 1, 2,…q} is a one
to one mapping. f is called graceful mapping if the induced mapping f^+: E(G) → {1, 2,…q}defined by f^+(uv) = | f(u) - f(v)| is also a one to one , onto mapping.

2.2 Super Vertex Graceful Labelling
Let G = (V, E) be a simple graph with p vertices and q edges. Let f:V(G) →P is a one to one mapping. f is called super vertex graceful mapping if the induced mapping f^*: E(G) →Q defined by f^+(uv) = f(u) + f(v) is also a bijection, where

$$P = \begin{cases} \pm 1, \ \pm 2, \ldots \pm \frac{p}{2} & if\ p\ is\ even \\ 0, \pm 1, \pm 2, \ldots \pm \frac{p-1}{2} & if\ p\ is\ odd \end{cases}$$

$$Q = \begin{cases} \pm 1, \ \pm 2, \ldots \pm \frac{q}{2} & if\ q\ is\ even \\ 0, \pm 1, \pm 2, \ldots \pm \frac{q-1}{2} & if\ q\ is\ odd \end{cases}$$

Examples

Consider the cycle C_5 with vertices$\{v_1, v_2, \ldots v_5\}$. Since both the order and size of C_5 are 5. Let $P = \{-2, -1, 0, 1, 2\} = Q$. Let f : V(C_5)→ P such that $f(v_1) = -2; f(v_2) = 0; f(v_3) = 2; f(v_4) = -1; f(v_5) = 1$. Let e_i be the edge connecting the vertices v_i and v_{i+1}, i=1, 2, 3,4 and $e_5$be the edge connecting v_5 and v_0. Then the mapping $f^+(e_1) = -2; f^+(e_2) = 2; f^+(e_1) = -2; f^+(e_3) = 1; f^+(e_5) = -1$ with "f" make C_5 as super vertex graceful graph.

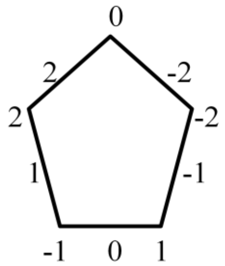

Fig. 1 A SVG of order 5 and size 5

As an another example, the following graph is also a super vertex graceful, where P = {-3, -2, -1, 1, 2, 3 } and Q = {-4, -3, -2, -1, 1, 2, 3, 4 }

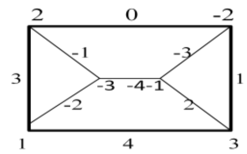

Fig. 2 A SVG of order 6 and size 8

3. RESULTS
3.1. Lemma
Let G be a graph with p vertices and q edges. If p is even, then max(Q) = p-1; min (Q) = -p+1.

Proof: If p is even, $P = \{\pm 1, \pm 2, \ldots \pm \frac{p-2}{2}, \pm \frac{p}{2}\}$. ∴ Max(Q) = $\frac{p}{2} + \frac{p-2}{2} = p - 1$, Min(Q) = $-\frac{p}{2} - \frac{p-2}{2} = -p + 1$.

3.2 Theorem
Let G be a graph with p vertices. If G is super vertex graceful then the size of G is utmost 2p-1 if p is even and 2p-3 if p is odd.

Proof:

Case(i)Let p is even. Then G has atmost$\frac{p(p-1)}{2}$ edges. In this case $P = \left\{\pm 1, \pm 2, \ldots \pm \frac{p-2}{2}, \pm \frac{p}{2}\right\}$. Then by the lemma 3.1,the set Q has maximum value $\frac{p}{2} + \frac{p-2}{2} = p - 1$ and minimum value –(p-1). Hence |G| has atmost 2(p+1)+1 = 2p-1, elements including the edge with label 0.

Case(ii) Let p be odd. Then $P = \{0, \pm 1, \pm 2, \ldots \pm \frac{p-1}{2}\}$. Then Q has maximum value p-2 and minimum value –(p-2). Hence |G| has atmost 2(p-2) +1 = 2p-3 elements. Hence the theorem.

Example

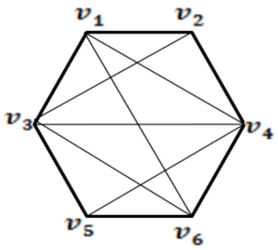

Fig. 3 A graph of order 6 and size 12

Here q =2p-1, p = 6 (even).Consider the graph given in fig.3 with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then by the definition of super vertex graceful map $P = \{-3, -2, -1, 1, 2, 3\}$. The size of G is 12. Then $Q = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$. If $fv3=3$and $fv4=2$, then $f+v3,v4=3+2=5$. Hence the induced map f^+ can induce only the values 1,2, 3, 4, 5, -1, -2, -3, -4, -5. The remaining 2 edges cannot be labelled.

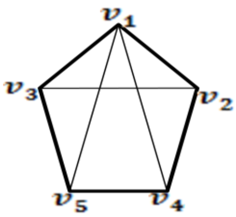

Fig. 4 A graph of order 5 and size 8

If q =2p-3, p = 5(odd). Then $P = \{-2, -1, 0, 1, 2\}$. The size of the graph is $Q = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. If $f(v_1) = 1$and $f(v_2) = 2$, then $f^+(v_1, v_2) = 1 + 2 = 3$. The labels 4, -4 cannot be induced by the map f^+. ∴the graph is not super vertex graceful.

3.3 Lemma

$\frac{n(n-1)}{2}$is even when n is a multiple of 4.

Proof:

Let us prove this lemma by induction principle on n.If n =4, $\frac{n(n-1)}{2} = \frac{4x3}{2} = 6$ which is even. Let n =k, be a multiple of 4 and $\frac{k(k-1)}{2}$ is an even number.ie$\frac{k(k-1)}{2} = 2m \Longrightarrow k^2 - k = 4m$. $\therefore k^2 = 4m + k$. For n = k+1, $\frac{(k+1)k}{2} = \frac{k^2+k}{2} = \frac{4m+k+k}{2} = \frac{4m+2k}{2} = 2m + k$which is also an even number.Hence by induction principle, it can be concluded that $\frac{n(n-1)}{2}$ is even for all multiples of 4.

3.4 Lemma

For$(i)\ n \geq 5, 2n - 1 < \frac{n(n-1)}{2}$; $(ii) 2n - 3 < \frac{n(n-1)}{2}$.

Proof:

We prove that (i), then (ii) is obvious. We use induction on n to prove (i). If n =5, 2n-1 =9, $\frac{n(n-1)}{2}$= 10. So, let $2k - 1 < \frac{k(k-1)}{2}$. We prove $2(k+1) - 1 < \frac{(k+1)k}{2}$.

Now 2(k+1)-1= 2k+2-1 = (2k-1)+2 $< \frac{k(k-1)}{2} + 2 = \frac{k^2-k+4}{2} < \frac{k^2+k}{2} = \frac{k(k+1)}{2}$.

3.5 Theorem

Let G = (V, E) be a super vertex graceful graph with even number of vertices and even number of edges. Then there is no vertex u in V, such that deg(u) = |V| -1.

Proof:

If possible assume that there is u in V such that deg(u) = |V| - 1. Also assume that the vertex u is labelled with integer $i \leq \frac{|V|}{2}$. Let f(u) = $i \leq \frac{|V|}{2}$. Since G is super vertex graceful graph, there must be a vertex v in V such that v is labelled with the integer –i i.ef(v) = -i. Moreover the vertex u is adjacent to every other vertex in G. In particular u is adjacent to v. Hence the induced map f^+ gives us that f^+(uv) = f (u) + f (v) = 0. Since G has even number of edges,0 $\notin Q$.This is a contradiction. Hence there is no u in V such that deg(u) = |V| - 1.

3.6 Lemma

The complete graphs K_2, K_3 are super vertex graceful graphs.

Proof:

Let V(K_2) = {u,v},and E(K_2) =e. P = {-1,1} and Q = {0}.K_2 is the graph with one edge and two vertices u and v. The edge e is incident with the vertices u and v. f is a map from V → P and if $f(u) = -1, f(v) = 1$ and f^+(e) =f^+(uv) = f (u) + f (v) = 1+(-1) = 0. Hence K_2 is super vertex graceful.

Let V(K_3) = {x,y,z},E(K_3) = {e_1, e_2,e_3}.Here P = {-1,0,1} and Q = {-1,0,1}.f is a map from V → P and if $f(x) = -1, f(y) = 1, f(z) = 0$. The edge e_1 is incident with x and y, $e_2$is incident with y and zand $e_3$is incident with x and z. Thenf^+(e_1) =f^+(xy) = f (x) + f (y) = 1+(-1) = 0,f^+(e_2) =f^+(yz) = f (y) + f (z) = 1+ 0 = 1;f^+(e_3) =f^+(xz) = f (x) + f (z) = 1+ 0 = 1.Hence K_3 is super vertex graceful.

3.7 Theorem

The complete graphs K_{4n}; n = 1, 2, 3… is not a super vertex graceful graph.

Proof:

From the theorem 3.5, it can be concluded that the number of edges is K_{4n} is even for all values of n. Hence K_{4n} has even number of vertices and even number of edges. Also, for every $u \in |V(K_{4n})| - 1$. Therefore from the above theorem, K_{4n} cannot be super vertex graceful.

Example

Consider the graph G = $K_{4n,}$ n = 3.Then the vertex set $V = \{v_1, v_2, \ldots, v_{12}\}$ and hence$P = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$. The number of edges is 66. Then$Q = \{\pm 1, \pm 2, \ldots \pm 33\}$. In $K_n d(v_i) = n - 1\ for\ all\ i$. If $f(v_k) = j\ for\ j \in P$. The vertex v_k is adjacent to the remaining 11 vertices of V For some r, $f(v_r) = -j$. Then the induced map $f^+(v_k, v_r) = f(v_k) + f(v_r) = j - j = 0 \notin Q$. ∴is not super vertex graceful.

3.8 Theorem

Complete bipartite graph $K_{(1,2n)}$ is a super vertex graceful for all values of n.

Proof:

Let G = {V, E, f, f^+} be a complete bipartite graph. The order and size of $K_{(1,2n)}$ is 2n +1 and 2n. Let V = {V_1,V_2} where V_1 and V_2 are the partitions of V . The complete bipartite graph is obtained by joining the vertices V_1 with vertices of V_2. f is a bijective mapping from V to P and f^+ is bijective map from E to Q as follows: f^+(u, v)= f(u)+ f(v), where P and Q are the vertex label and edge label sets respectively and are given by P = {0, ±1, ±2, ⋯ , ±n} and Q = {±1, ±2, ⋯ , ±n}. Let the set P be partitioned into two sets P_1 and P_2 such that P = P_1∪ P_2 and P_1 ∩ P_2 = Φ. i.e. P_1 = 0; P_2 = {±1, ±2, …, ±n}. Now we define the induced map f^+ by

f^+(u, v_i)= f(u)+ f(v_i) = {-n,-(n-1), …-1,1, 2, …,(n-1), n}.Therefore the map $f$is a SVG map for all values of 'n' and hence $K_{(1,2n)}$ is a SVG.

3.9 Theorem

The complete bipartite graph $K_{2,2n+1}$ is super vertex graceful.

Proof:

Let G = {V, E, f, f^+} be a bipartite graph. The order of $K_{(2,2n-1)}$ is 2n +1 and size is 4n − 2. The vertex label set P = {0, ±1, ±2, ⋯ , ±n} and the edge label set Q = {0, ±1, ±2, ⋯ , ±2n − 1}. The vertex set is partitioned into two sets V_1 and V_2 where V_1 = {x_1,x_2} V_2 = {x_3,x_4, ⋯ ,x_{2n+1}}. The map fi : Vi→ Pi is a bijective map. The map f^+ : E→Q is also a bijective map defined as f^+(u, v)= f(u)+ f(v). Here P_1 = {±n} and P_2 = {0, ±1, ±2 ⋯± (n − 1)}. Then G is a SVG graph with f^+ as SVG mapping.

4. DISCUSSION

The labelling of graphs under super vertex graceful map depends on the order and size. Similar analysis can be done for other mapping and some particular graphs.

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