Two New Parametric Generalized R–Norm Fuzzy Information Measures

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ABSTRACT
The paper introduces two new parametric generalizations of one of existing $R-$norm fuzzy information measures with the proof of their validity. In addition, particular cases and important properties of the proposed measures are discussed. A numerical example is given to establish the similarity between the proposed $R-$norm fuzzy information measures with one of the existing $R-$norm fuzzy information measures. Further, a comparison among them is shown with the help of a table and graph.

General Terms
Fuzzy Set Theory, Fuzzy Information Theory.

Keywords
Fuzziness, Fuzzy set, Fuzzy measure of information, $R-$norm fuzzy information measure.

1. INTRODUCTION
Entropy is very important for measuring uncertain information. Shannon [12] was first to use the word "entropy" to measure the uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P=(p_1,p_2,...p_n)$ in an experiment. The information contained in this experiment is given by

$$H(P)=-\sum_{i=1}^{n} p_i \log p_i \tag{1}$$

which is the well known Shannon [12] entropy.

A fuzzy set A defined on a universe of discourse X is given as Zadeh [13]:

$$A=\left\{\langle x,\mu_A(x)\rangle / x\in X\right\}$$

where $\mu_A : X\to[0,1]$ is the membership function of A. The membership value $\mu_A(x)$ describes the degree of the belongingness of $x\in X$ in A. When $\mu_A(x)$ is valued in $\{0,1\}$, it is the characteristic function of a crisp (i.e., non-fuzzy) set.

A fuzzy set A^* is the sharpened version of A if the following conditions are satisfied:

$$\mu_{A^*}(x_i)\le \mu_A(x_i)\text{, if } \mu_A(x_i)\le 0.5;\ \forall i$$

and $\quad \mu_{A^*}(x_i)\ge \mu_A(x_i)$, if $\mu_A(x_i)\ge 0.5;\ \forall i$.

Zadeh [13] gave some notions related to fuzzy sets, which are used in the discussion, as follows:

(1) Compliment: $\overline{A}=$ Compliment of $A \Leftrightarrow \mu_{\overline{A}}(x)=1-\mu_A(x)$ for all $x\in X$.

(2) Union: $A\cup B=$ Union of A and B $\Leftrightarrow \mu_{A\cup B}(x)=\max\{\mu_A(x),\mu_B(x)\}$ for all $x\in X$.

(3) Intersection: $A\cap B=$ Intersection of A and B $\Leftrightarrow \mu_{A\cap B}(x)=\min\{\mu_A(x),\mu_B(x)\}$ for all $x\in X$.

Fuzziness is found in our decision, in our language and in the way we process information. Fuzziness, a feature of uncertainty, results from the lack of sharp difference of being or not being an element is the member of a set. A measure of fuzziness is called the fuzzy entropy, first mentioned by Zadeh [14]. Fuzzy entropy is an important tool for measuring fuzzy information. It has wide applications in the area of pattern recognition, image processing, speech recognition, medical diagnosis, decision making etc.

De Luca and Termini [3] introduced the measure of fuzzy entropy corresponding to Shannon [12] entropy given in (1) as

$$H(A)=-\sum_{i=1}^{n}[\mu_A(\mathrm{x_i})\log\mu_A(\mathrm{x_i})+(1-\mu_A(\mathrm{x_i}))\log(1-\mu_A(\mathrm{x_i}))] \tag{2}$$

satisfying the following essential properties:

(P1) $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x_i)=0$ or 1 for all x_i.

(P2) $H(A)$ is maximum if and only if A is most fuzzy set, i.e. $\mu_A(x_i)=0.5$ for all x_i.

(P3) $H(A)=H(A^*)$, where A^* is sharpened version of A.

(P4) $H(A)=H(\overline{A})$, where $\overline{A}$ is the complement of A.

Later on Bhandari and Pal [1] defined the following exponential fuzzy entropy corresponding to Pal and Pal [11] exponential entropy as

$$E(A)=\frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}[\mu_A(x_i)e^{(1-\mu_A(x_i))}+(1-\mu_A(x_i))e^{\mu_A(x_i)}-1] \quad (3)$$

Boekee & Lubbe [2] defined and studied $R-$norm information measure of the distribution P for $R\in\mathbb{R}^{+}$ as given by

$$H_R(P)=\frac{R}{R-1}\left[1-(\sum_{i=1}^{n}p_i^R)^{\frac{1}{R}}\right]; R>0, R\neq 1. \quad (4)$$

Analogous to (4) Hooda [4] proposed the following $R-$norm fuzzy measure of information

$$H_R(A)=\frac{R}{R-1}\sum_{i=1}^{n}\left[1-(\mu_A^R(x_i)+(1-\mu_A(x_i))^R)^{\frac{1}{R}}\right];$$
$R(>0)\neq 1$. (5)

Further, from the significant studies it is noted that Hooda and Bajaj [7] and Hooda and Jain [8] provide the generalization of $R-$norm fuzzy information measure (5) corresponding to generalized $R-$norm information measures proposed by Hooda and Ram [5] and Hooda and Sharma [6] respectively.

Kumar [9] generalized the measure (4) and gave $R-$norm measure of information of order α which is

$$H_R^{\alpha}(P)=\frac{R}{R-\alpha}\left[1-\left(\sum_{i=1}^{n}p_i^{\frac{R}{\alpha}}\right)^{\frac{\alpha}{R}}\right];\ 0<\alpha\leq 1,\ R(>0)\neq 1. \quad (6)$$

Further, Kumar and Choudhary [10] generalized the measure (4) and gave the $R-$norm information measure of degree m as

$$H_R^{m}(P)=\frac{R-m+1}{R-m}\left[1-\left(\sum_{i=1}^{n}p_i^{R-m+1}\right)^{\frac{1}{R-m+1}}\right]; R-m+1>0,$$
$R\neq m$, $R,m>0(\neq 1)$. (7)

Inspired by the above-mentioned work, the paper proposes two new generalized $R-$norm fuzzy information measures and provides the study of the essential properties of these measures in order to check their validity. The remainder of the paper is organized as follows: In section 2, two new parametric generalizations of $R-$norm measure of fuzzy information are defined and the essential properties are proved to check their authenticity. In section 3, a numerical example is given to make a comparison between the proposed generalized $R-$norm measures of fuzzy information with $R-$norm fuzzy information measure (5). Finally, the concluding remarks are given in section 4.

2. PARAMETRIC GENERALIZED R – NORM FUZZY INFORMATION MEASURES

Corresponding to measures given by Kumar [9] & Kumar and Choudhary [10] in (6) and (7), the following generalized $R-$norm fuzzy information measures are proposed:

$$H_R^{\alpha}(A)=\frac{R}{R-\alpha}\sum_{i=1}^{n}\left[1-(\mu_A^{\frac{R}{\alpha}}(x_i)+(1-\mu_A(x_i))^{\frac{R}{\alpha}})^{\frac{\alpha}{R}}\right];$$
$0<\alpha\leq 1$, $R(>0)\neq 1$. (8)

$$H_R^{m}(A)=\frac{R-m+1}{R-m}\sum_{i=1}^{n}\left[1-(\mu_A^{R-m+1}(x_i)+(1-\mu_A(x_i))^{R-m+1})^{\frac{1}{R-m+1}}\right]$$
; $R-m+1>0$, $R\neq m$, $R,m>0(\neq 1)$. (9)

Theorem 1: The generalized $R-$norm fuzzy information measures given by (8) and (9) are valid measures of fuzzy information.

Proof: P1 (Sharpness): The measures (8) and (9) clearly satisfy the property **P1**, i.e., $H_R^{\alpha}(A)=0$ if and only if A is non-fuzzy set or crisp set and $H_R^{m}(A)=0$ if and only if A is non-fuzzy set or crisp set.

P2 (Maximality): To verify that the proposed measure is concave; the different values of $H_R^{\alpha}(A)$ firstly for a fixed value of R and different values of α, secondly for a fixed value of α and different values of R are computed.

Case 1: Let us assume a particular value of $R=0.5$ and different values of α. The computed values of $H_R^{\alpha}(A)$ for $R=0.5$ using (8) and different values of α are given in Table 1.

Table 1 Different values of $H_R^{\alpha}(A)$ for $R=0.5$ and for different values of α

α	$\mu_A(x_i)$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	$H_{0.5}^{0.1}(A)$	0.0	0.1250	0.2498	0.3725	0.4812	0.7179	0.4812	0.3725	0.2498	0.1250	0.0
0.3	$H_{0.5}^{0.3}(A)$	0.0	0.2155	0.3832	0.5054	0.5802	0.6053	0.5802	0.5054	0.3832	0.2155	0.0
0.4	$H_{0.5}^{0.4}(A)$	0.0	0.2705	0.4437	0.5588	0.6254	0.6472	0.6254	0.5588	0.4437	0.2705	0.0
0.7	$H_{0.5}^{0.7}(A)$	0.0	0.4319	0.6124	0.7204	0.7797	0.7988	0.7797	0.7204	0.6124	0.4319	0.0
1.0	$H_{0.5}^{1.0}(A)$	0.0	0.6	0.8	0.9165	0.9798	1	0.9798	0.9165	0.8	0.6	0.0

Case 2: Let us assume a particular value of $\alpha = 0.7$ and different values of R. The computed values of $H_R^\alpha(A)$ for $\alpha = 0.7$ using (8) and different values of R are given in Table 2.

Table 2 Different values of $H_R^\alpha(A)$ for $\alpha = 0.7$ and for different values of R

R	$\mu_A(x_i)$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	$H_{0.1}^{0.7}(A)$	0.0	6.8071	8.6641	9.7356	10.3152	10.5	10.3152	9.7356	8.6641	6.8071	0.0
0.2	$H_{0.2}^{0.7}(A)$	0.0	1.2087	1.5379	1.7276	1.8301	1.8627	1.8301	1.7276	1.5379	1.2087	0.0
0.4	$H_{0.4}^{0.7}(A)$	0.0	0.5275	0.7174	0.8289	0.8896	0.9090	0.8896	0.8289	0.7174	0.5275	0.0
0.6	$H_{0.6}^{0.7}(A)$	0.0	0.3698	0.5467	0.6552	0.7153	0.7348	0.7153	0.6552	0.5467	0.3698	0.0
0.8	$H_{0.8}^{0.7}(A)$	0.0	0.2911	0.4652	0.5783	0.6429	0.6640	0.6429	0.5783	0.4652	0.2911	0.0

Thus $H_R^\alpha(A)$ is a concave function with respect to R and α and its maximum value exists at $\mu_A(x_i) = 0.5$. Similarly, the concavity of $H_R^m(A)$ can be proved. Thus, $H_R^\alpha(A)$ and $H_R^m(A)$ are concave functions and their maximum values exist at $\mu_A(x_i) = 0.5$, i.e., maxima exists if and only if A is the fuzziest set. Thus, (8) and (9) satisfy the property **P2**.

P3 (Resolution): Since $H_R^\alpha(A)$ and $H_R^m(A)$ are increasing functions of $\mu_A(x_i)$ in the range $[0, 0.5)$ and decreasing function in the range $(0.5, 1]$, therefore

$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_R^\alpha(A^*) \le H_R^\alpha(A)$ in $[0, 0.5)$

and $\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow H_R^\alpha(A^*) \ge H_R^\alpha(A)$ in $(0.5, 1]$.

Taking the above equations together, it comes

$$H_R^\alpha(A^*) \le H_R^\alpha(A).$$

Similarly, $H_R^m(A^*) \le H_R^m(A)$.

P4 (Symmetry): From the definition of $H_R^\alpha(A)$ and $H_R^m(A)$ and with $\mu_{\overline{A}}(x_i) = 1 - \mu_A(x_i)$, it is obvious that $H_R^\alpha(\overline{A}) = H_R^\alpha(A)$ and $H_R^m(\overline{A}) = H_R^m(A)$.

Hence $H_R^\alpha(A)$ and $H_R^m(A)$ satisfy all the four properties (P1) to (P4) of measures of fuzzy information, therefore these are valid measures of fuzzy information. The measure (8) can be called as the generalized $\alpha - R -$ norm fuzzy information measure and (9) as the generalized $R -$ norm fuzzy information measure of type R and degree m.

Limiting and Particular Cases:

(i) When $\alpha = 1$ and $m = 1$, (8) and (9) reduce to $H_R(A)$.

(ii) When $\alpha = 1$, $m = 1$ and $R \to 1$, (8) and (9) reduce to $H(A)$.

(iii) When $\alpha = 1$, $m = 1$ and $R \to \infty$, (8) and (9) reduce to $\sum_{i=1}^{n} [1 - \max\{\mu_A(x_i), 1 - \mu_A(x_i)\}]$.

Theorem 2: For $A, B \in FS(X)$

$H_R^\alpha(A \cup B) + H_R^\alpha(A \cap B) = H_R^\alpha(A) + H_R^\alpha(B)$.

Proof: Let $X_1 = \{x / x \in X, \mu_A(x_i) \ge \mu_B(x_i)\}$ (10)

$$X_2 = \{x / x \in X, \mu_A(x_i) < \mu_B(x_i)\} \quad (11)$$

Where $\mu_A(x_i)$ and $\mu_B(x_i)$ be the fuzzy membership functions of A and B respectively.

$$H_R^\alpha(A \cup B) = \frac{R}{R-\alpha} \sum_{i=1}^{n} \left[1 - (\mu_{A \cup B}^{\alpha}(x_i) + (1 - \mu_{A \cup B}(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}} \right]$$

$$= \frac{R}{R-\alpha} \left[\sum_{X_1} [1 - (\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}}] + \sum_{X_2} [1 - (\mu_B^{\alpha}(x_i) + (1 - \mu_B(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}}] \right] \quad (12)$$

$$H_R^\alpha(A \cap B) = \frac{R}{R-\alpha} \sum_{i=1}^{n} \left[1 - (\mu_{A \cap B}^{\alpha}(x_i) + (1 - \mu_{A \cap B}(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}} \right]$$

$$= \frac{R}{R-\alpha} \left[\sum_{X_1} [1 - (\mu_B^{\alpha}(x_i) + (1 - \mu_B(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}}] + \sum_{X_2} [1 - (\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha})^{\frac{R}{\alpha}\frac{\alpha}{R}}] \right] \quad (13)$$

Adding (12) and (13) gives

$$H_R^\alpha(A \cup B) + H_R^\alpha(A \cap B) = H_R^\alpha(A) + H_R^\alpha(B).$$

Hence, the proof of theorem.

In Particular: For $A \in FS(X)$, $A \in FS(X)$ where $\overline{A}$ the complement of fuzzy set A, it gets

$$H_R^\alpha(A) = H_R^\alpha(\overline{A}) = H_R^\alpha(A \cup \overline{A}) = H_R^\alpha(A \cap \overline{A}) \quad (14)$$

Theorem 3: For $A, B \in FS(X)$, $H_R^m(A \cup B) + H_R^m(A \cap B) = H_R^m(A) + H_R^m(B)$.

Proof: Clearly the result can be proved on similar lines as in theorem 2.

In Particular: For $A \in FS(X)$, $A \in FS(X)$ where $\overline{A}$ the complement of fuzzy set A, it gets

$$H_R^m(A) = H_R^m(\overline{A}) = H_R^m(A \cup \overline{A}) = H_R^m(A \cap \overline{A}). \qquad (15)$$

3. NUMERICAL EXAMPLE

Example: Let $A = \{(x_i, \mu_A(x_i)) / x_i \in X\}$ be a fuzzy set in $X = (x_1, x_2, ..., x_n)$. For any real number *n*, from the operation of power of a fuzzy set:

$$A^n = \{(x_i, [\mu_A(x_i)]^n) / x_i \in X\}.$$

Let us assume a standard fuzzy set A on $X = (x_1, x_2, ..., x_n)$ defined as:

$$A = (0.2, 0.3, 0.2, 0.4, 0.5)$$

By taking into consideration the characterization of linguistic variables, A is considered as "LARGE" on X. Using the above operation:

$A^{1/2}$ may be treated as "More or less LARGE"; A^2 may be treated as "Very LARGE";

A^3 may be treated as "Quite very LARGE"; A^4 may be treated as "Very very LARGE".

Next these fuzzy sets are used to compare the above proposed fuzzy entropy measures with one of $R-$norm fuzzy entropy measures. From the point of logical consideration, it may be mentioned that the entropies of fuzzy sets are required to follow the following order pattern:

$$H_R(A^{1/2}) > H_R(A) > H_R(A^2) > H_R(A^3) > H_R(A^4) \qquad (16)$$

The calculated numerical values of three fuzzy information measures for these cases are given in Tables 3, 4 and 5 below:

Table 3 Numerical values of the $R-$norm fuzzy entropy measure H_R

H_R	$A^{1/2}$	A	A^2	A^3	A^4
$R=0.1$	328.9041	258.0100	179.0883	118.0176	79.5181
$R=0.5$	4.8587	4.4963	2.9554	1.8314	1.1392
$R=0.9$	4.4643	3.1244	1.7781	0.9322	0.4897
$R=2$	2.7415	2.3220	1.0367	0.4409	0.1940
$R=5$	2.3658	1.8854	0.7242	0.2900	0.1242

For any particular value of $\alpha = 0.7$,

Table 4 Numerical values of the $R-$norm fuzzy entropy measure H_R^α

H_R^α	$A^{1/2}$	A	A^2	A^3	A^4
$R=0.1$	51.2045	47.8754	33.5312	22.6947	15.6592
$R=0.5$	3.8610	3.5240	2.1409	1.2058	0.6798
$R=0.9$	3.0597	3.3325	1.3683	0.5984	0.3062
$R=2$	2.5640	2.1088	0.8703	0.3548	0.1527
$R=5$	2.2744	1.7958	0.6744	0.2698	0.1156

For any particular value of $m = 0.7$,

Table 5 Numerical values of $R-$norm fuzzy entropy measure H_R^m

H_R^m	$A^{1/2}$	A	A^2	A^3	A^4
$R=0.1$	5.9352	5.5234	3.7551	2.4330	1.5886
$R=0.5$	3.6502	3.3124	1.9470	1.0613	0.5777
$R=0.9$	3.1255	2.7618	1.4353	0.6890	0.3335
$R=2$	2.6660	2.2309	0.9625	0.4010	0.1743
$R=5$	2.3492	1.8684	0.7144	0.2859	0.1225

The numerical values given in Table 3 reveal that the $R-$norm fuzzy entropy measure $H_R(A)$ satisfies (16). The results presented in Tables 4 & 5 clarify that the proposed new generalized $R-$norm fuzzy entropy measures $H_R^\alpha(A)$ and $H_R^m(A)$ satisfy the same:

$$H_R^\alpha(A^{1/2}) > H_R^\alpha(A) > H_R^\alpha(A^2) > H_R^\alpha(A^3) > H_R^\alpha(A^4) \qquad (17)$$

and

$$H_R^m(A^{1/2}) > H_R^m(A) > H_R^m(A^2) > H_R^m(A^3) > H_R^m(A^4). \qquad (18)$$

Thus, the behaviour of new generalized R – norm fuzzy entropy measures $H_R^{\alpha}(A)$ and $H_R^{m}(A)$ is also consistent for the structured linguistic variables. Table 6 and Figure 1 display a declining trend in the numerical values of three entropy measures corresponding to the logical order of fuzzy sets.[1]

Table 6 Calculated numerical values of proposed fuzzy entropy measures and the existing one

$R = 0.5$	H_R (Existing)	H_R^{α} (Proposed)	H_R^{m} (Proposed)
$A^{1/2}$	4.8587	3.8610	3.6502
A	4.4963	3.5240	3.3124
A^2	2.9554	2.1409	1.9470
A^3	1.8314	1.2058	1.0613
A^4	1.1392	0.6798	0.5777

Figure 1 also depicts the similarity between the proposed entropy measures and the existing one.

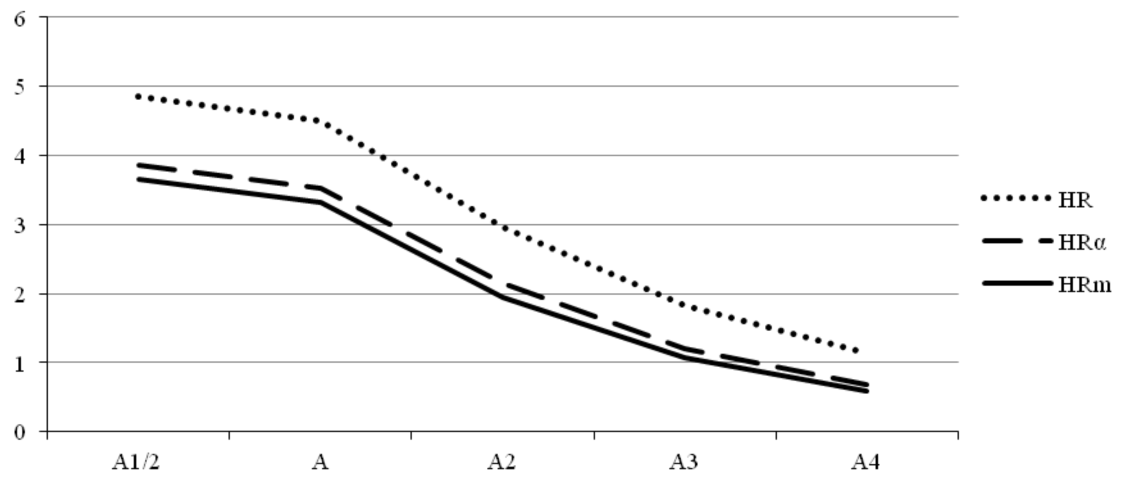

Figure 1: Comparison of numerical values of H_R , H_R^{α} and H_R^{m}

An inequality among two proposed measures of fuzzy information and the existing one may also be seen from Figure 1, that is: $H_R \geq H_R^{\alpha} \geq H_R^{m}$.

4. CONCLUSION

In the paper, two new parametric generalizations of one of existing R – norm fuzzy information measures are proposed with the proof of their validity. The proposed generalized fuzzy measures of information are valid measures which reduce to the known measure on substituting the particular values of parameters. Some of the interesting properties of these measures have also been studied. The given numerical example proves the similarity of proposed generalized measures of fuzzy information with $H_R(A)$. Thus, the proposed measures are more flexible measures from the application point of view.

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