Characterization of the Inverse Exponential-Type Distribution Based on Recurrence Relations for Dual Generalized Order Statistics

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ABSTRACT
In this paper, new recurrence relations satisfied by the single and product moments using moment generating function of dual generalized order statistics from inverse exponential-type distribution are established. Recurrence relations for single and product moments of reversed order statistics and lower record value are obtained as special cases. Further, using a recurrence relation for single moments we obtain characterization of inverse exponential-type distribution.

Keywords
Dual generalized order statistics, moment generating function, recurrence relations, single moments, product moments, inverse exponential form and characterization.

1. INTRODUCTION
The importance of generalized order statistics (gos) due to inclusion special cases which have been used in statistical research such as order statistics and record value, see [1]. Let \( F(\cdot) \) be absolutely continuous distribution function with probability density function \( f(\cdot) \). Further, let \( n \in \mathbb{N} \), \( k \geq 0 \), \( n \geq 2 \), \( m = (m_1, m_2, \ldots, m_n) \in \mathbb{R}^{n-1} \). \( M_r = \sum_{i=1}^{n} m_i \), such that \( \gamma_r = k + n - r + M_r > 0 \), for all \( r \in \{1, 2, \ldots, n-1\} \). Then \( X(n,n,m,k) \), \( r=1,2,\ldots,n \) are called gos if their joint pdf is expressed from (1) as

\[
F^{-1}(1) = x_1 > x_2 > \cdots > x_n > F^{-1}(0).
\]

The marginal pdf of r-th dgos, \( X'(r,n,m,k) \) is

\[
f_X'(r,n,m,k)(x) = \frac{C_{r-1}}{\Gamma(r)} \Gamma(x-1)^{r-1} f(x) g_m^{r-1}(F(x)).
\]

The joint pdf of \( X'(r,n,m,k) \) and \( X'(s,n,m,k) \), \( 1 \leq r \leq s \leq n \), \( x > y \) is expressed from (1) as

\[
f_{X'(r,n,m,k)X'(s,n,m,k)}(x,y) = \frac{C_{r-1}}{\Gamma(r) \Gamma(s-r)} \Gamma(x-1)^{r-1} f(x) g_m^{r-1}(F(x))
\]

\[
\times [ h_m(F(y)) - h_m(F(x))]^{s-r-1} \Gamma(y)^{s-r-1} f(y),
\]

where

\[
\Gamma(r) = \int_0^\infty t^{r-1} \exp(-t) dt,
\]

\[
C_{r-1} = \prod_{i=1}^{r-1} \gamma_i,
\]

[2] represents the concept of dual generalized order statistics (dgos) as follows:

Let \( n \in \mathbb{N} \), \( k \geq 1 \), be the parameters such that

\[
\gamma_r = k + (n-r)(m+1) > 0, \text{ for all } l, r, n.
\]
\[ h_n(x) = \begin{cases} \frac{-1}{m+1} x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1. \end{cases} \]
and
\[ g_n(x) = h_n(x) - h_n(1), x \in [0,1). \]

Since \( X \) \((0,n,m,k) \to 0 \) then \( X \) \((n+1,n,m,k) \to 0 \). If 
\( m = 0, k = 1 \), then \( X \) \((r,n,m,k) \) reduces to the 
\((n-r+1)-th\) reversed order statistics, \( (X_{n-r+1}) \) from the 
sample \( X_1, X_2, \ldots, X_n \) and when \( m = -1 \), then
\( X \) \((r,n,m,k) \) reduces to the \( k \)-lower record value [3].

[1] dealt with generalized order statistics from kumaraswamy
distribution and its characterization. Recurrence relations
for moment generating functions of order statistics are
established by [4]. [2] established moments of lower
generalized order statistics from exponentiated Pareto
distribution and its characterization. Recurrence relations
for single and product moments of dual generalized order
statistics from the inverse Weibull distribution are derived
by [5], [6] have established recurrence relations for moments
dual generalized order statistics from exponentiated Weibull
distribution. [7] obtained recurrence relations for moment
and conditional moment generating functions of gos based
on random samples drawn from a population whose distribution
is a member of a doubly truncated class of distributions.

Consider the cumulative distribution function (cdf) is:
\[ F(x) = \exp[\lambda(x)], x \geq 0, \quad (4) \]
where \( \lambda(x) \) is a non-negative, continuous, monotone
decreasing, differentiable function of \( x \) such that \( \lambda(x) \to 0 \)
as \( x \to \infty \) and \( \lambda(x) \to \infty \) as \( x \to 0^+\). This family contains
many distributions such as inverse Weibull distribution,
inverse exponential distribution and inverse Rayleigh
distribution.

The probability density function (pdf) is given by:
\[ f(x) = -\lambda'(x)\exp[-\lambda(x)], x \geq 0. \quad (5) \]

Therefore, from (4) and (5), we have
\[ F(x) = -\frac{1}{\lambda(x)} f(x). \quad (6) \]

The following table gives some distributions with proper
choice of \( \lambda(x) \) as examples:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>cdf</th>
<th>( \lambda(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>inverse Weibull</td>
<td>\exp(- (ax)^\beta)</td>
<td>\alpha^{-\beta} x, x &gt; 0, \alpha, \beta &gt; 0.</td>
</tr>
<tr>
<td>inverse exponential</td>
<td>\exp(- (ax)^{\beta})</td>
<td>\frac{1}{ax}, x &gt; 0, \beta = 1.</td>
</tr>
<tr>
<td>inverse Rayleigh</td>
<td>\exp(- (ax)^{\beta})</td>
<td>\frac{1}{ax}, x &gt; 0, \alpha, \beta = 2.</td>
</tr>
</tbody>
</table>

2. CHARACTERIZATION BASED ON
RECURSION RELATIONS FOR
SINGLE MOMENT GENERATING
FUNCTIONS OF DGOS

The single moment generating function of dgos can be
obtained, for \( a \geq 1 \), from Using (2), we have when \( m = -1 \)
\[ M_{(r,n,m,k)}(t) = E \left[ \frac{\alpha^{(o)}}{\gamma(r)^n} \right] \]
\[ = \frac{C_{(n+1)}}{\Gamma(r)^n} \int_0^\infty \left( F(x) \right)^{\gamma-1} f(x) g_n^{-1}(F(x)) \, dx, \]
for simplicity \( X_{(r,n,m,k)} = x \).

Relation 1 Let \( X \) be a random variable, then for integers \( a \)
such that \( a \geq 1 \), the following recurrence relation is satisfied
if \( X \) has cdf (4).

\[ M_{(r,n,m,k)}(t) - M_{(r-1,n,m,k)}(t) = \frac{a t}{\gamma(r)^n} \left[ \frac{X_{(r-1,n,m,k)}(t)}{\lambda(X_{(r,n,m,k)})} \right]. \]

Proof: Using (7) and (2), we have
\[ M_{(r,n,m,k)}(t) = \frac{C_{(n+1)}}{\Gamma(r)^n} \int_0^\infty \left( F(x) \right)^{\gamma-1} f(x) g_n^{-1}(F(x)) \, dx, \]
integrating by parts, we get
\[ M_{(r,n,m,k)}(t) - M_{(r-1,n,m,k)}(t) \]
\[ = \frac{a t C_{(n+1)}}{\gamma(r)^n} \int_0^\infty \left( F(x) \right)^{\gamma-1} g_n^{-1}(F(x)) \, dx, \]
upon using (1.6) in (2.4), we obtain
\[ M_{(r,n,m,k)}(t) - M_{(r-1,n,m,k)}(t) \]
\[ \frac{a t C^{-\gamma}_r}{\gamma r \Gamma(r)} \int x^{e-1} e^{\alpha F} (F(x))^{r-1} g_n^{-1}(F(x)) dx , \]

so, we have the result.

Conversely, if the characterizing condition (8) holds, then from (2) and (6), we have

\[ C^{-\gamma}_r \int x^{e-1} e^{\alpha F} (F(x))^{r-1} g_n^{-1}(F(x)) dx = 0, \]

integrating the first integral on the right hand side of Equation (11) by parts, we get

\[ \frac{a t C^{-\gamma}_r}{\gamma r \Gamma(r)} \int x^{e-1} e^{\alpha F} (F(x))^{r-1} g_n^{-1}(F(x)) dx + \frac{a t C^{-\gamma}_r}{\gamma r \Gamma(r)} \int x^{e-1} e^{\alpha F} (F(x))^{r-1} f(x) g_n^{-1}(F(x)) dx = 0. \]

which reduces to

\[ \frac{a t C^{-\gamma}_r}{\gamma r \Gamma(r)} \int x^{e-1} e^{\alpha F} (F(x))^{r-1} g_n^{-1}(F(x)) dx = 0. \]

Now applying a generalization of the Muntz-Szasz Theorem to (13) [8], we get

\[ F(x) = - \frac{1}{\lambda} f(x), x \geq 0. \]

3. SPECIAL CASES

(1) By differentiating both sides of Condition (8) with respect to \( t \) and then setting \( t = 0 \), we obtain the following recurrence relation for single moment of dgos:

\[ \mu^{(s)}_{r,n,m,k} - \mu^{(s)}_{r,n,m,k} = \frac{a}{\gamma_l} E \left[ \frac{X^{(e)}_{r,n,m,k}}{\lambda (X_{r,n,m,k})} \right] \]

(14)

where

\[ \mu^{(s)}_{r,n,m,k} = E (X^{(s)}_{r,n,m,k}). \]

(2) Putting \( m = 0, k = 1 \) in (8), we obtain the recurrence relations of reversed order statistics as follows:

\[ M^{(s)}_{r,n}(t) - M^{(s)}_{r,n+1}(t) = \frac{a t}{n - r + 1} E \left[ \frac{X^{(e)}_{r,n+1}}{\lambda (X_{r,n+1})} \right], \]

(15)

where \( \gamma_l = n - r + 1, X_{r,n,m,k} = X_r \).

(3) Setting \( m = -1, k = 1 \) in (8), we get recurrence relation for single moment of lower record values in the form:

\[ M^{(s)}_{L_{1,r}}(t) - M^{(s)}_{L_{1,r+1}}(t) = \frac{a t}{r - 1} E \left[ \frac{X^{(e)}_{L_{1,r+1}}}{\lambda (X_{L_{1,r+1}})} \right], \]

(16)

where \( \gamma_l = 1, X_{r,n,m,k} = X_{L_{1,r}} \).

4. CHARACTERIZATION BASED ON RECURRENCE RELATIONS FOR CONDITIONAL MOMENT GENERATING FUNCTIONS OF DGOS

On using (3), the conditional distribution function of \( X(s,n,m,k) \) given \( X(s,n,m,k) \), is given by:

\[ f \left[ X^{(e)}_{r,n,m,k} \right] X_{r,n,m,k} = y = \frac{C_{r,i}}{C_{r,j}F(s-r)} \left[ F(y) \right]^{r-1} \int [h_n(F(x)) - h_n(F(y))]^{e-1} f(x) dx. \]

(17)

Relation 2 Let \( X \) be a random variable, \( r,s \) be two integers such that \( l = l, s, n, m \) and \( k \) be real numbers such that \( m \geq 1, k \geq 1 \). Then for integers \( a \) such that \( a \geq 1 \), the following recurrence relation is satisfied if \( X \) has the cdf (4).
\[ M_{X(y)}(t) - M_{X(y)}(t) = E\left[ e^{\frac{a}{\gamma_r}} X_{(r,a,m,k)}(t) \right] = \frac{aC_{r,s} [F(y)]^{-\gamma_r s}}{C_{r,s} s \Gamma(s-r)} \]

Proof Using (17), we get
\[ M_{X(y)}(t) = E\left[ e^{\frac{a}{\gamma_r}} X_{(r,a,m,k)}(t) \right] = \frac{C_{r,s} [F(y)]^{-\gamma_r s}}{C_{r,s} \Gamma(s-r)} \]

Integrate (19) by parts, we get
\[ \int_{0}^{t} e^{\alpha x} [h_n(F(x)) - h_m(F(y))]^{-1} f(x) [F(x)]^{-1} dx \]

Substituting (6) and (20) in (19), we get
\[ \int_{0}^{t} e^{\alpha x} [h_n(F(x)) - h_m(F(y))]^{-1} f(x) [F(x)]^{-1} dx \]

So, we can rewrite Equation (21) in the form
\[ M_{X(y)}(t) - M_{X(y)}(t) = E\left[ e^{\frac{a}{\gamma_r}} X_{(r,a,m,k)}(t) \right] = \frac{aC_{r,s} [F(y)]^{-\gamma_r s}}{C_{r,s} s \Gamma(s-r)} \]

then, after some simplifications, we obtain
\[ \int_{0}^{t} e^{\alpha x} [h_n(F(x)) - h_m(F(y))]^{-1} f(x) [F(x)]^{-1} dx = 0 \]
Applying Muntz-Szasz theorem to (25) [8], we get
\[ F(x) = \frac{1}{\lambda}f(x). \] (26)

4.1 Special cases

(1) By differentiating both sides of Condition (18) with respect to \( t \) and then setting \( t = 0 \), we obtain the following recurrence relation for single moment of dgos:
\[
E\left[ X_{r,n}^a \mid X_{(r,n)} \right] = y
\]
\[-E\left[ X_{(r,n)}^a \mid X_{r,n} \right] = y
\]
\[= \frac{a}{\gamma_r} E\left[ \frac{X_{r,n}^{a-1}X_{(r,n)}^a}{\lambda \left( X_{r,n} \right)} \right] X_{r,n} = y. \] (27)

(2) Putting \( m = 0, k = 1 \) in (18), we obtain the recurrence relations of reversed order statistics as follows:
\[
M_{x_{r,n}^a \mid x_{r,n}^a} (t \mid y) - M_{x_{(r,n)}^a \mid x_{r,n}^a} (t \mid y) = \frac{a}{(n-s+1)} E\left[ \frac{X_{r,n}^{a-1}X_{(r,n)}^a}{\lambda \left( X_{r,n} \right)} \right] X_{r,n} = y.
\]
\[
E\left[ X_{r,n}^a \mid X_{r,n} = y \right] - E\left[ X_{(r,n)}^a \mid X_{r,n} = y \right] = \frac{a}{(n-s+1)} E\left[ \frac{X_{r,n}^{a-1}X_{(r,n)}^a}{\lambda \left( X_{r,n} \right)} \right] X_{r,n} = y.
\]

(3) Setting \( m = -1, k = 1 \) in (18), we get recurrence relation for single moment of lower order values in the form:
\[
M_{x_{L(r)}^a \mid x_{L(r)}^a} (t \mid y) - M_{x_{(L(r))}^a \mid x_{L(r)}^a} (t \mid y) = \frac{a}{(n-s+1)} E\left[ \frac{X_{L(r)}^{a-1}X_{(L(r))}^a}{\lambda \left( X_{L(r)} \right)} \right] X_{L(r)} = y.
\]
\[
E\left[ X_{L(r)}^a \mid X_{L(r)} = y \right] - E\left[ X_{(L(r))}^a \mid X_{L(r)} = y \right] = \frac{a}{(n-s+1)} E\left[ \frac{X_{L(r)}^{a-1}X_{(L(r))}^a}{\lambda \left( X_{L(r)} \right)} \right] X_{L(r)} = y.
\]

5. CHARACTERIZATION BASED ON PRODUCT MOMENT FOR DGOS

Lemma For \( 1 \leq r < s \leq n-1, n \geq 2 \) and \( k = 1, 2, \ldots \)
\[
E(X^r (r,n,m,k)X^s (s,n,m,k))
\]
\[-E(X^r (r,n,m,k)X^s (s-1,n,m,k))
\]
\[= \frac{-jc_{r+1}}{\gamma_r \Gamma(r) \Gamma(s-r)} \int_0^\infty y^{r-i} [F(x)]^m f(x) g^{s-r-1}_n [F(x)]
\]
\[\times [h_n(F(x)) - h_n(F(y))]^{s-r-1} [F(y)]^{r-1} \left[ \frac{-1}{\lambda \left( y \right)} \right] f(y) dy dx. \] (30)

Relation 3 Let \( X \) be a random variable, then for integers \( a \) such that \( a \geq 1 \), the following recurrence relation is satisfied if \( X \) has cdf (4).
\[
E(X^r (r,n,m,k)X^s (s,n,m,k))
\]
\[-E(X^r (r,n,m,k)X^s (s-1,n,m,k))
\]
\[= \frac{j}{\gamma_r} E\left[ X^j \right] (r,n,m,k)X^j (s,n,m,k)]. \]

Proof: From (30), we get
\[
E[X^r (r,n,m,k)X^s (s,n,m,k)]
\]
\[-E(X^r (r,n,m,k)X^s (s-1,n,m,k))
\]
\[= \frac{-jc_{r+1}}{\gamma_r \Gamma(r) \Gamma(s-r)} \int_0^\infty y^{r-i} [F(x)]^m f(x) g^{s-r-1}_n [F(x)]
\]
\[\times [h_n(F(x)) - h_n(F(y))]^{s-r-1} [F(y)]^{r-1} \left[ \frac{-1}{\lambda \left( y \right)} \right] f(y) dy dx. \] (32)

and hence (31) obtained.

Conversely, if the characterizing condition (4.2) holds, then from (3) and (6), we have
\[
\frac{C_{r+1}}{\Gamma(r) \Gamma(s-r)} \int_0^\infty y^{r-i} [F(x)]^m f(x) g^{s-r}_m [F(x)]
\]
\[\times [h_n(F(x)) - h_n(F(y))]^{s-r} f(y) dy dx
\]
\[= \frac{C_{r+2}}{\Gamma(r) \Gamma(s-r)} \int_0^\infty y^{r-i} [F(x)]^m f(x) g^{s-r+1}_n [F(x)]
\]
\[\times [h_n(F(x)) - h_n(F(y))]^{s-r+1} f(y) dy dx
\]

\[= \frac{C_{r+2}}{\Gamma(r) \Gamma(s-r)} \int_0^\infty y^{r-i} [F(x)]^m f(x) g^{s-r+1}_n [F(x)]
\]
\[\times [h_n(F(x)) - h_n(F(y))]^{s-r+1} f(y) dy dx
\]
\[ j \frac{C_{r+1}}{\gamma_r} \int_{s-r}^{s} \left\{ \frac{x^r}{\lambda(x)} \right\}^{s-1} f(x) \, dx \]
\[ \times g_{m,n}^{-1}(F(x)) [h_n(F(y)) - h_n(F(x))]^{-1} \]
\[ = j E \left[ \frac{X_i^r}{\lambda(y)} \right] (r,n,-1,1) X_i^{-1} (s,n,-1,1). \quad (37) \]

6. THE INVERSE WEIBULL DISTRIBUTION
The distribution function of the inverse Weibull distribution is given by:
\[ F(x \mid \theta) = \exp[-(ax)^\beta], \quad x > 0, \quad (38) \]
where \( \theta = (\alpha, \beta), \, \alpha > 0, \, \beta > 0 \) and
\[ \lambda(x) = \frac{\alpha^\beta}{x^{\beta+1}}, \quad (39) \]
From Equations (14), (27) and (31), we have the following relations:
\[ \mu_{x,m:k}^{(i)} - \mu_{x-1,m:k}^{(i)} = -\frac{\alpha a^\beta}{\beta y_{r}}, \quad (40) \]
\[ E \left[ X_{(r,m,k)}^o \right] \mid X_{(r,m,k)} = y \]
\[ = E \left[ X_{(r-1,m,k)}^o \right] \mid X_{(r,m,k)} = y \]
\[ = \frac{-\alpha a^\beta}{\beta y_{r}} E \left[ X_{(r,m,k)}^o \right] \mid X_{(r,m,k)} = y \].
\[ E\left[ X_i^o(r,n,m,k) \right] X_i^\beta (s,n,m,k) \]
\[ = \frac{-\alpha a^\beta}{\beta y_{r}} E\left[ X_i^o(r,n,m,k) \right] X_i^\beta (s,n,m,k). \quad (42) \]

7. CONCLUSION
The recurrence relations for single and product moments based on moment generating function of dual generalized order statistics from the inverse exponential-type distribution are obtained. The recurrence relation for the single moments is used to characterize the inverse exponential-type distribution. Recurrence relations for single and product moments of reversed order statistics and Power record values are deduced as special cases. All results can be applied to many inverse distribution such as inverse exponential, inverse Rayleigh and inverse Weibull.

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9. REFERENCES


