Convexity of Minimal Dominating and Total Dominating Functions of Corona Product Graph of a Cycle with a Complete Graph

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ABSTRACT
‘Domination in graphs’ has been studied extensively and at present it is an emerging area of research in graph theory. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [1,2].

Product of graphs occurs naturally in discrete mathematics as tools in combinatorial constructions. They give rise to an important classes of graphs and deep structural problems. In this paper we study the dominating and total dominating functions of corona product graph of a cycle with a complete graph.

Keywords
Corona Product, Cycle, Complete Graph, Dominating function, Total dominating function.

Subject Classification: 68R10

1. INTRODUCTION
Domination Theory is an important branch of Graph Theory that has many applications in Engineering, Communication Networks, mobile computing, resource allocation, telecommunication and many others. Allan, R.B. and Laskar, R.[3], Cockayne, E.J. and Hedetniemi, S.T. [4] have studied various domination parameters of graphs.

Recently, dominating functions in domination theory have received much attention. The concepts of total dominating functions and minimal total dominating functions are introduced by Cockayne et al. [5]. Jeelani Begum, S. [6] has studied some total dominating functions of Quadratic Residue Cayley graphs.

Frucht and Harary [7] introduced a new product on two graphs G1 and G2, called corona product denoted by G1  G2. The object is to construct a new and simple operation on two graphs G1 and G2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G1 and of G2.

The authors have studied some dominating functions of corona product graph of a cycle with a complete graph [8] and published papers on minimal dominating functions, some variations of Y – dominating functions and Y – total dominating functions [9,10,11].

In this paper we discuss the convexity of minimal dominating and total dominating functions of corona product graph of a cycle with a complete graph.

2. CORONA PRODUCT OF Cn AND Km
The corona product of a cycle Cn with a complete graph Km is a graph obtained by taking one copy of a n – vertex graph Cn and n copies of Km and then joining the ith vertex of Cn to every vertex of ith copy of Km and it is denoted by Cn  Km.

3. CONVEXITY OF MINIMAL DOMINATING FUNCTIONS
A study of convexity and minimality of dominating functions (MDFs) are given in Cockayne et al.[12] and Yu[13]. Rejikumar [14] developed a necessary and sufficient condition for the convex combination of MDF to be again a MDF. Jeelani Begum [6] studied convexity of MDFs of Quadratic Residue Cayley Graphs.

In this section we discuss the convexity of minimal dominating functions of the corona product graph G = Cn  Km. First we define the convex combination of functions and prove some results on the convexity of MDFs of G.

Definition: Let G = (V, E) be a graph. Let f and g be two functions from V to [0, 1] and λ e (0, 1). Then the function h : V  [0, 1] defined by h(v) = λ f(v) + (1 − λ) g(v) is called a convex combination of f and g.

Definition: Let G = (V, E) be a graph. A subset D of V is said to be a dominating set (DS) of G if every vertex in V − D is adjacent to some vertex in D.

A dominating set D is called a minimal dominating set (MDS) if no proper subset of D is a dominating set of G.

Definition: The domination number of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by γ(G).

Definition: Let G = (V, E) be a graph. A function f : V → [0,1] is called a dominating function (DF) of G if f(N[v]) = ∑ f(u) ≥ 1, for each v e V.

Definition: Let f and g be functions from V to [0,1]. We define f < g if f(u) ≤ g(u) for all u e V, with strict inequality for at least one vertex u e V.
A dominating function \( f \) of \( G \) is called a \textbf{minimal dominating function} (MDF) if for all \( g < f \), \( g \) is not a dominating function.

\textbf{Theorem 3.1:} Let \( D_1 \) and \( D_2 \) be two MDSs of \( G = C_n \odot K_m \).
Let \( f_1: V \rightarrow [0, 1] \) and \( f_2: V \rightarrow [0, 1] \) be defined by
\[
f_1(v) = \begin{cases} 1, & \text{if } v \in D_1, \\ 0, & \text{otherwise}. \end{cases}
\]
and \( f_2(v) = \begin{cases} 1, & \text{if } v \in D_2, \\ 0, & \text{otherwise}. \end{cases}\)

Then the convex combination of \( f_1 \) and \( f_2 \) becomes a MDF of \( G = C_n \odot K_m \).

\textbf{Proof:} Let \( D_1 \) and \( D_2 \) be two MDSs of \( G \). Let \( f_1, f_2 \) be two functions defined as in the hypothesis.

These functions become MDFs of \( G = C_n \odot K_m \) [8].

Let \( h(v) = \alpha f_1(v) + \beta f_2(v) \), where \( \alpha + \beta = 1 \) and \( 0 < \alpha < 1, 0 < \beta < 1 \).

\textbf{Case 1:} Suppose \( D_1 \cap D_2 \neq \emptyset \).

For \( v \in V \), the possible values of \( h(v) \) are
\[
h(v) = \begin{cases} \alpha, & \text{if } v \in D_1 - D_2, \\ \beta, & \text{if } v \in D_2 - D_1, \\ \alpha + \beta, & \text{if } v \in D_1 \cap D_2, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( \sum_{u \in N[v]} h(u) = s \alpha + t \beta \), if \( s \)-vertices of \( D_1 \) and \( t \)-vertices of \( D_2 \) are in \( N[v] \).

So \( \sum_{u \in N[v]} h(u) \geq 1 \), \( \forall \ v \in V \).

This implies that \( h \) is a DF.

Now we check for the minimality of \( h \).

Define \( g : V \rightarrow [0, 1] \) by
\[
g(v) = \begin{cases} r, & \text{if } v = v_i \in D_1, \\ \alpha, & \text{if } v \in (D_1 \cap D_2) - \{v_i\}, \\ \alpha + \beta, & \text{if } v \in D_1 - D_2, \\ \beta, & \text{if } v \in D_2 - D_1, \\ 0, & \text{otherwise}. \end{cases}
\]

where \( 0 < r < 1 \).

Since strict inequality holds at the vertex \( v_i \in V \), it follows that \( g < h \).

Then \( \sum_{u \in N[v]} g(u) = r + \beta < \alpha + \beta = 1 \). for the vertices in the \( i \)th copy of \( K_m \) in \( G \).
4. CONVEXITY OF MINIMAL TOTAL DOMINATING FUNCTIONS

The concepts of total dominating functions (TDFs) and minimal total dominating functions (MTDFs) are introduced by Cockayne et al. [15]. A study of convexity and minimality of TDFs are given in Cockayne et al. [12, 15]. Cockayne et al. [12] obtained a necessary and sufficient condition for the convex combination of two minimal total dominating functions to be again a minimal total dominating function.

The authors have studied minimal total dominating functions of $G = C_n \oplus K_m$ [16]. In this section we consider minimal total dominating functions of corona product graph $G = C_n \oplus K_m$ and discuss their convexity.

Definition: Let $G(V, E)$ be a graph without isolated vertices. A subset $T$ of $V$ is called a total dominating set (TDS) if every vertex in $V$ is adjacent to at least one vertex in $T$.

Definition: The minimum cardinality of a MTDS of $G$ is called a total domination number of $G$ and is denoted by $\gamma_t(G)$.

Definition: Let $G(V, E)$ be a graph. A function $f: V \rightarrow \{0, 1\}$ is called a total dominating function (TDF) of $G$ if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$, for each $v \in V$.

Definition: Let $f$ and $g$ be functions from $V$ to $\{0, 1\}$. We define $f < g$ if $f(u) < g(u)$ for all $u \in V$, with strict inequality for at least one vertex $u \in V$.

A TDF $f$ of $G$ is called a minimal total dominating function (MTDF) if for all $f < g$, $g$ is not a TDF.

Theorem 4.1: Let $T_1$ and $T_2$ be two MTDSs of $G = C_n \oplus K_m$. Let $f_1: V \rightarrow \{0, 1\}$ and $f_2: V \rightarrow \{0, 1\}$ be defined by

$$f_1(v) = \begin{cases} 1, & \text{if } v \in T_1, \\ 0, & \text{otherwise}. \end{cases}$$

and

$$f_2(v) = \begin{cases} 1, & \text{if } v \in T_2, \\ 0, & \text{otherwise}. \end{cases}$$

Then the convex combination of $f_1$ and $f_2$ becomes a MTDF of $G = C_n \oplus K_m$.

Proof: Let $T_1$ and $T_2$ be two MTDSs of $G$. Let $f_1, f_2$ be two functions defined as in the hypothesis.

These functions become MTDFS of $G = C_n \oplus K_m$ [8].

Let $h(v) = \alpha f_1(v) + \beta f_2(v)$, where $\alpha + \beta = 1$ and $0 < \alpha < 1$, $0 < \beta < 1$.

Case 1: Suppose $T_1 \cap T_2 \neq \emptyset$.

Then for $v \in V$, the possible values of $h(v)$ are

$$h(v) = \begin{cases} \alpha, & \text{if } v \in T_1 - T_2, \\ \beta, & \text{if } v \in T_2 - T_1, \\ \alpha + \beta, & \text{if } v \in T_1 \cap T_2, \\ 0, & \text{otherwise}. \end{cases}$$

Now

$$\sum_{u \in N(v)} h(u) = s\alpha + t\beta, \quad \text{if } s \text{-vertices of } T_1 \text{ and } t \text{-vertices of } T_2 \text{ are in } N(v).$$

Therefore

$$\sum_{u \in N(v)} h(u) \geq 1, \quad \forall v \in V.$$  

This implies $h$ is a TDF.

Now we check for the minimality of $h$.

Define $g : V \rightarrow \{0, 1\}$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_i \in T_1 \cap T_2, \\ \alpha + \beta, & \text{if } v \in (T_1 \cap T_2) - \{v_i\}, \\ \alpha, & \text{if } v \in T_1 - T_2, \\ \beta, & \text{if } v \in T_2 - T_1, \\ 0, & \text{otherwise}. \end{cases}$$

where $0 < r < 1$.

Since strict inequality holds at the vertex $v_i \in T_1 \cap T_2$, it follows that $g < h$.

Then

$$\sum_{u \in N(v)} g(u) \leq \sum_{v \in V} g(v) = r < 1,$$

for the vertices in the $i^{th}$ copy of $K_m$ in $G$.

So $g$ is not a TDF.

Since $g$ is taken arbitrarily, it follows that there exists no $g < h$ such that $g$ is a TDF.

Thus $h$ is a MTDF.

Case 2: Suppose $T_1 \cap T_2 = \emptyset$.

Then for $v \in V$, the possible values of $h(v)$ are
\[ h(v) = \begin{cases} 
\alpha, & \text{if } v \in T_1, \\
\beta, & \text{if } v \in T_2, \\
0, & \text{otherwise.} 
\end{cases} \]

Now
\[ \sum_{u \in N(v)} h(u) = s\alpha + t\beta, \]
if \( s \) - vertices of \( T_1 \) and \( t \) - vertices of \( T_2 \) are in \( N(v) \).

Therefore
\[ \sum_{u \in N(v)} h(u) \geq 1, \quad \forall \ v \in V. \]

This implies that \( h \) is a TDF.

Now we check for the minimality of \( h \).

Define \( g : V \rightarrow [0, 1] \) by
\[ g(v) = \begin{cases} 
\tau, & \text{if } v = v_1 \in T_1, \\
\alpha, & \text{if } v \in T_1 \setminus \{v_1\}, \\
\beta, & \text{if } v \in T_2, \\
0, & \text{otherwise.} 
\end{cases} \]

where \( 0 < \tau < \alpha \).

Since strict inequality holds at the vertex \( v_1 \in T_1 \), it follows that \( g < h \).

Then \( \sum_{u \in N(v)} g(u) \) is defined as:
\[ \begin{cases} 
\tau + \beta, & \text{if } v \in i^a \text{ copy of } K_m \text{ in } G, \\
\alpha + \beta, & \text{if } s - \text{vertices of } T_1 \text{ and } t - \text{vertices of } T_2 \text{ are in } N(v), \\
\alpha + 2\beta, & \text{if } s - \text{vertices of } T_1 \text{ and } t - \text{vertices of } T_2 \text{ are in } N(v). 
\end{cases} \]

This implies that \( \sum_{u \in N(v)} g(u) = \tau + \beta < \alpha + \beta = 1 \) for the vertices in the \( i^{th} \) of \( K_m \) in \( G \).

So \( g \) is not a TDF.

Since \( g \) is taken arbitrarily, it follows that there exists no \( g < h \) such that \( g \) is a TDF.

Thus \( h \) is a MTDF. \( \blacksquare \)

5. CONCLUSION
It is interesting to study the convexity of minimal dominating and total dominating functions of corona product graph of a cycle with a complete graph. This work gives the scope for an extensive study of dominating functions in general of these graphs.

6. REFERENCES