Generalized Coupled Fibonacci Sequences of rth Order and their Properties

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ABSTRACT
Among numerical sequences, the Fibonacci numbers have achieved a kind of celebrity status with such fabulous properties, it is no wonder that the Fibonacci numbers stand out as a kind of super sequence. Fibonacci numbers have been studied in many different forms for centuries and the literature on the subject is consequently, incredibly vast. The Fibonacci sequence has been generalized in a number of ways. The purpose of this paper is to demonstrate many of the properties of Coupled Fibonacci sequences of rth order, which can be stated and proved for a much more general class.

Subject Classification: **11B39, 11B37, 11B99.**

Keyword
Fibonacci sequences, Coupled Fibonacci sequences

1. INTRODUCTION
Fibonacci numbers have been studied in many different forms for centuries and the literature on the subject is consequently, incredibly vast. The Fibonacci sequence has been generalized in a number of ways. In 1985, Attanasov [1] introduced a new view of generalized Fibonacci sequences by taking a pair of sequences $\{\alpha_i\}_{i=0}^{\infty}$and $\{\beta_i\}_{i=0}^{\infty}$which can be generated by famous Fibonacci formula and gave various identities involving Fibonacci sequence called the coupled Fibonacci sequences. With some initial values a, b, c and d four different schemes [2] were defined as below $\forall\, n \geq 0$.

The four schemes are following

For $\alpha_0 = a, \alpha_1 = b, \beta_0 = c, \beta_1 = d$

First scheme is

$$\alpha_{n+2} = \beta_{n+1} + \beta_n \text{ and } \beta_{n+2} = \alpha_{n+1} + \alpha_n, \forall\, n \geq 0 \qquad (1.1)$$

Second scheme is

$$\alpha_{n+2} = \alpha_{n+1} + \beta_n \text{ and } \beta_{n+2} = \beta_{n+1} + \alpha_n, \forall\, n \geq 0 \qquad (1.2)$$

Third scheme is

$$\alpha_{n+2} = \beta_{n+1} + \alpha_n \text{ and } \beta_{n+2} = \alpha_{n+1} + \beta_n, \forall\, n \geq 0 \qquad (1.3)$$

Fourth scheme is

$$\alpha_{n+2} = \alpha_{n+1} + \alpha_n \text{ and } \beta_{n+2} = \beta_{n+1} + \beta_n, \forall\, n \geq 0 \qquad (1.4)$$

In recent years many authors have been generalized coupled Fibonacci sequences in third, fourth and fifth order. In this paper, the coupled Fibonacci sequence has been generalized for rth order with some new interesting properties.

2. COUPLED FIBONACCI SEQUENCES OF rth ORDER
Attanasov,K.T.[1][2][3] introduced new generalized coupled Fibonacci sequences.

In the same way, here we let two sequences with initial conditions

$$\alpha_0 = a_{0,}\alpha_1 = a_{1,}\alpha_2 = a_{2\ldots.}\alpha_{r-1} = a_{r-1}$$

$$and\ \beta_0 = b_{0,},\beta_1 = b_{1,}\beta_2 = b_{2\ldots.}\beta_{r-1} = b_{r-1}$$

Then generalized coupled Fibonacci sequences defined by

$$\alpha_{n+r} = \sum_{i=0}^{r-1} \beta_{n+i} = \beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3} + \cdots \beta_{n+r-2} + \beta_{n+r-1} \qquad (2.1)$$

$$\beta_{n+r} = \sum_{i=0}^{r-1} \alpha_{n+i} = \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \cdots \alpha_{n+r-2} + \alpha_{n+r-1} \qquad (2.2)$$

First few terms of the sequences are given below

n	$\boldsymbol{\alpha_{n+r}}$	$\boldsymbol{\beta_{n+r}}$
0	$b_0 + b_1 + b_2 + \cdots b_{r-1}$	$a_0 + a_1 + a_2 + \cdots a_{r-1}$
1	$(a_0 + a_1 + a_2 + a_3 + \cdots + a_{r-1}) + (b_1 + b_2 + b_3 + \cdots + b_{r-1})$	$(a_1 + a_2 + a_3 + \cdots + a_{r-1}) + (b_0 + b_1 + b_2 + \cdots + b_{r-1})$
2	$(a_0 + 2a_1 + 2a_2 + 2a_3 + \cdots + 2a_{r-1}) + (b_0 + b_1 + 2b_2 + \cdots + b_{r-1})$	$(a_0 + a_1 + 2a_2 + 2a_3 + \cdots + 2a_{r-1}) + (b_0 + 2b_1 + 2b_2 + \cdots + 2b_{r-1})$
3	$(2a_0 + 3a_1 + 4a_2 + \cdots + 4a_{r-1}) + (2b_0 + 3b_1 + 3b_2 + 4b_3 + \cdots + 4b_{r-1})$	$(2a_0 + 3a_1 + 3a_2 + 4a_3 + \cdots + 4a_{r-1}) + (2b_0 + 3b_1 + 4b_2 + \cdots + 4b_{r-1})$
4	$(4a_0 + 5a_1 + 7a_2 + 8a_3 + 8a_4 + \cdots + 8a_{r-1}) + (4b_0 + 6b_1 + 7b_2 + 7b_3 + 8b_4 + \cdots + 8b_{r-1})$	$(4a_0 + 5a_1 + 7a_2 + 7a_3 + 8a_4 + \cdots + 8a_{r-1}) + (4b_0 + 6b_1 + 7b_2 + 8b_3 + 8b_4 + \cdots + 8b_{r-1})$

3. MAIN RESULTS

Many authors have been constructed the identities of coupled Fibonacci sequences under addition. In this section many of other fabulous properties can likewise be established by induction method.

Theorem 3.1.For every integer $n \geq 0, r \geq 0$

$$\alpha_{n(r+1)} + \beta_0 = \beta_{n(r+1)} + \alpha_0$$

Proof: If n = 0, then result is true because

$$\alpha_0 + \beta_0 = \beta_0 + \alpha_0$$

Assume that the result is true for some integer $n \geq 1$

Now by Equation

$\alpha_{n+r} = \sum_{i=0}^{r-1} \beta_{n+i}$ and $\beta_{n+r} = \sum_{i=0}^{r-1} \alpha_{n+i}$

Using induction method for $n = n + 1$, then

$$\alpha_{(n+1)(r+1)} + \beta_0 = [\beta_{n(r+1)+r} + \beta_{n(r+1)+r-1} + \beta_{n(r+1)+r-2} + \cdots + \beta_{n(r+1)+1}] + \beta_0$$

$$= [\alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \alpha_{n(r+1)+(r-3)} + \cdots + \alpha_{n(r+1)+1} + \alpha_{n(r+1)}] + [\beta_{n(r+1)+r-1} + \beta_{n(r+1)+r-2} + \beta_{n(r+1)+r-3} + \cdots + \beta_{n(r+1)+1}] + \beta_0$$ (By equation 2.1)

$$= [\alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \alpha_{n(r+1)+(r-3)} + \cdots + \alpha_{n(r+1)+1}] + [\beta_{n(r+1)+(r-1)} + \beta_{n(r+1)+(r-2)} + \beta_{n(r+1)+(r-3)} + \cdots + \beta_{n(r+1)+1}] + \alpha_{n(r+1)} + \beta_0$$

$$\alpha_{(n+1)(r+1)} + \beta_0 = [\alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \alpha_{n(r+1)+(r-3)} + \cdots + \alpha_{n(r+1)+1}] + [\beta_{n(r+1)+(r-1)} + \beta_{n(r+1)+(r-2)} + \beta_{n(r+1)+(r-3)} + \cdots + \beta_{n(r+1)+1}] + \beta_{n(r+1)} + \alpha_0$$ (By induction hypothesis)

$$= [\alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \alpha_{n(r+1)+(r-3)} + \cdots + \alpha_{n(r+1)+1}] + \alpha_{n(r+1)+r} + \alpha_0$$

$$= \beta_{n(r+1)+(r+1)} + \alpha_0$$

$$= \beta_{(n+1)(r+1)} + \alpha_0$$

Hence the result is true for n = n+1.

Theorem 3.2 For every integer $n \geq 0, r \geq 0$

$$\alpha_{n(r+1)+1} + \beta_1 = \beta_{n(r+1)+1} + \alpha_1$$

Proof: If n = 0, then result is true because

$$\alpha_1 + \beta_1 = \beta_1 + \alpha_1$$

Assume that the result is true for some integer $n \geq 1$

Now by Equation

$\alpha_{n+r} = \sum_{i=0}^{r-1} \beta_{n+i}$ and $\beta_{n+r} = \sum_{i=0}^{r-1} \alpha_{n+i}$

Using induction method for $n = n + 1$, then

$$\alpha_{(n+1)(r+1)+1} + \beta_1 = \alpha_{n(r+1)+(r+2)} + \beta_1$$

$$= [\beta_{n(r+1)+(r+1)} + \beta_{n(r+1)+r} + \beta_{n(r+1)+r-1} + \cdots + \beta_{n(r+1)+2}] + \beta_1$$

$$= [\alpha_{n(r+1)+r} + \alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \cdots + \alpha_{n(r+1)+2} + \alpha_{n(r+1)+1}] + [\beta_{n(r+1)+r} + \beta_{n(r+1)+(r-1)} + \beta_{n(r+1)+(r-2)} + \cdots + \beta_{n(r+1)+2}] + \beta_1$$

(By equation 2.1)

$$=[\alpha_{n(r+1)+r} + \alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \cdots + \alpha_{n(r+1)+2}] + [\beta_{n(r+1)+r} + \beta_{n(r+1)+(r-1)} + \beta_{n(r+1)+(r-2)} + \cdots + \beta_{n(r+1)+2}] + \alpha_{n(r+1)+1} + \beta_1$$

$$\alpha_{(n+1)(r+1)+1} + \beta_1 = [\alpha_{n(r+1)+r} + \alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \cdots + \alpha_{n(r+1)+2}] + [\beta_{n(r+1)+r} + \beta_{n(r+1)+(r-1)} + \beta_{n(r+1)+(r-2)} + \cdots + \beta_{n(r+1)+2}] + \beta_{n(r+1)+1} + \alpha_1$$ (By induction hypothesis)

$$= [\alpha_{n(r+1)+r} + \alpha_{n(r+1)+(r-1)} + \alpha_{n(r+1)+(r-2)} + \cdots + \alpha_{n(r+1)+2}] + \alpha_{n(r+1)+(r+1)} + \alpha_1$$

$$= \beta_{n(r+1)+(r+1)+1} + \alpha_1$$

$$= \beta_{(n+1)(r+1)+1} + \alpha_1$$

Hence the result is true for n = n+1.

In the similar way, it could be proved for the following results by induction method.

Theorem 3.3 For every integer $n \geq 0, r \geq 0$

$$\alpha_{n(r+1)+2} + \beta_2 = \beta_{n(r+1)+2} + \alpha_2$$

Theorem 3.4 For every integer $n \geq 0, r \geq 0$

$$\alpha_{n(r+1)+3} + \beta_3 = \beta_{n(r+1)+3} + \alpha_3$$

Theorem 3.5 (General Theorem)

On the basis of above results, some important result

For every integer $n \geq 0, r \geq 0$ are

$$\alpha_{n(r+1)+p} + \beta_p = \beta_{n(r+1)+p} + \alpha_p \quad (3.1)$$

Where p is also some non negative integer.

Proof: If n = 0, then result is true because

$$\alpha_p + \beta_p = \beta_p + \alpha_p$$

We assume that the result is true for some integer $n \geq 1$

Now by Equation

$$\alpha_{n+r} = \sum_{i=0}^{r-1} \beta_{n+i} = \beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3} + \cdots \beta_{n+r-2} + \beta_{n+r-1}$$

$$\beta_{n+r} = \sum_{i=0}^{r-1} \alpha_{n+i} = \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \cdots \alpha_{n+r-2} + \alpha_{n+r-1}$$

Using induction method for $n = n + 1$, then

$$\alpha_{(n+1)(r+1)+p} + \beta_p = \alpha_{n(r+1)+(r+p+1)} + \beta_p$$

$$= [\beta_{n(r+1)+(r+p)} + \beta_{n(r+1)+(r+p-1)} + \beta_{n(r+1)+(r+p-2)} + \cdots + \beta_{n(r+1)+(p+1)}] + \beta_p$$

$= [\alpha_{n(r+1)+(r+p-1)} + \alpha_{n(r+1)+(r+p-2)} + \alpha_{n(r+1)+(r+p-3)} + \cdots + \alpha_{n(r+1)+p}] + [\beta_{n(r+1)+(r+p-1)} + \beta_{n(r+1)+(r+p-2)} + \beta_{n(r+1)+(r+p-3)} + \cdots + \beta_{n(r+1)+(p+1)}] + \beta_p$ (By equation 2.1)

$$=[\alpha_{n(r+1)+(r+p-1)} + \alpha_{n(r+1)+(r+p-2)} + \alpha_{n(r+1)+(r+p-3)} + \cdots + \alpha_{n(r+1)+(p+1)}] + [\beta_{n(r+1)+(r+p-1)} + \beta_{n(r+1)+(r+p-2)} + \beta_{n(r+1)+(r+p-3)} + \cdots + \beta_{n(r+1)+(p+1)}] + \alpha_{n(r+1)+p} + \beta_p$$

$$\alpha_{(n+1)(r+1)+p} + \beta_p = [\alpha_{n(r+1)+(r+p-1)} + \alpha_{n(r+1)+(r+p-2)} + \alpha_{n(r+1)+(r+p-3)} + \cdots + \alpha_{n(r+1)+(p+1)}] + [\beta_{n(r+1)+(r+p-1)} + \beta_{n(r+1)+(r+p-2)} + \beta_{n(r+1)+(r+p-3)} + \cdots + \beta_{n(r+1)+(p+1)}] + \beta_{n(r+1)+p} + \alpha_p$$

By induction hypothesis

$$= [\alpha_{n(r+1)+(r+p-1)} + \alpha_{n(r+1)+(r+p-2)} + \alpha_{n(r+1)+(r+p-3)} + \cdots + \alpha_{n(r+1)+(p+1)}] + \alpha_{n(r+1)+(r+p)} + \alpha_p$$

$$= \beta_{n(r+1)+(r+p)+1} + \alpha_p$$

$$= \beta_{(n+1)(r+1)+p} + \alpha_p$$

4. SPECIAL CASES

4.1. In this paper two Generalized Fibonacci sequence has been assumed. If r = 3 in the scheme 2.1 and 2.2 then the third order coupled sequences

$$\alpha_{n+3} = \beta_n + \beta_{n+1} + \beta_{n+2}$$

$$\beta_{n+3} = \alpha_n + \alpha_{n+1} + \alpha_{n+2}$$

and in similar way, results from above theorems which will be like as 3-order sequence

$(a)\alpha_{4n} + \beta_0 = \beta_{4n} + \alpha_0$

$(b)\alpha_{4n+1} + \beta_1 = \beta_{4n+1} + \alpha_1$

$(c)\alpha_{4n+2} + \beta_2 = \beta_{4n+2} + \alpha_2$

$(d)\alpha_{4n+3} + \beta_3 = \beta_{4n+3} + \alpha_3$

4.2. If r= 4 then sequence becomes

$$\alpha_{n+4} = \beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3}$$

$$\beta_{n+4} = \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3}$$

and results are

$(a)\alpha_{5n} + \beta_0 = \beta_{5n} + \alpha_0$

$(b)\alpha_{5n+1} + \beta_1 = \beta_{5n+1} + \alpha_1$

$(c)\alpha_{5n+2} + \beta_2 = \beta_{5n+2} + \alpha_2$

$(d)\alpha_{5n+3} + \beta_3 = \beta_{5n+3} + \alpha_3$

4.3. If r = 5, then sequence becomes

$$\alpha_{n+5} = \beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3} + \beta_{n+4}$$

$$\beta_{n+5} = \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \alpha_{n+3} + \alpha_{n+4}$$

and results are

$(a)\alpha_{6n} + \beta_0 = \beta_{6n} + \alpha_0$

$(b)\alpha_{6n+1} + \beta_1 = \beta_{6n+1} + \alpha_1$

$(c)\alpha_{6n+2} + \beta_2 = \beta_{6n+2} + \alpha_2$

$(d)\alpha_{6n+3} + \beta_3 = \beta_{6n+3} + \alpha_3$

In previous years these results has been proved by many mathematicians [6] with the help of same technique.

On the basis of these now we can construct the relations for different ordered coupled Fibonacci sequences.

5. CONCLUSION

There are many known identities for coupled Fibonacci sequences. This paper describes comparable identities of generalized coupled Fibonacci sequence of r^{th} order, the special cases for lower order were also discussed. Simply the ideas can be extended for generalized coupled Fibonacci sequences of r^{th} order with negative integers.

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