A New Perspective to the Sequences of t-Order

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ABSTRACT

In this paper, we consider two sequences of t-order $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by $\alpha_0 = a_1, \beta_0 = a_2, \alpha_1 = a_3$, $\beta_1 = a_4, ... \alpha_{t-1} = a_{2t-1}, \ \beta_{t-1} = a_{2t}$

$$\alpha_{n+t} = \sum_{i=0}^{t-1} \beta_{n+i}, \ \beta_{n+t} = \sum_{i=0}^{t-1} \alpha_{n+i}, \ n \geq 0,$$

where $a_1, a_2, \ldots, a_{2t-1}, a_{2t}$ are fixed real numbers and $t \in \mathbb{Z}^+ \backslash \{1\}$. Furthermore, some interesting properties of these sequences are given

Keywords

Sequences of t-order, integer function.

1. INTRODUCTION

In [1], the authors gave some identities involving the terms of two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by

$$\alpha_0 = a, \ \beta_0 = b, \alpha_1 = c, \ \beta_1 = d,$$

$$\alpha_{n+2} = \beta_{n+1} + \beta_n, \beta_{n+2} = \alpha_{n+1} + \alpha_n, n \geq 0, \qquad (1)$$

where a, b, c, and d are fixed real numbers.

For example, for $n \geq 0$, the authors obtained the following identities:

$$\alpha_{3k+2} = \sum_{i=0}^{3k} \beta_i + \beta_1, \ \beta_{6k+5} = \sum_{i=0}^{3k+2} \alpha_{2i} - \beta_0 + \alpha_1.$$

In [4], the authors considered the generalized recursive form of (1). In [2, 3, 7], the authors described new ideas for 3-Fibonacci sequences. In [8], the authors showed fundamental properties of 3-Fibonacci sequences. In [5], the authors gave some properties of two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ which have given initial values a, c, e, g and b, d, f, h (which are real numbers), and called 4-order sequences.

In [5], the authors obtained some interesting results for two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ which have given initial values a, c, e, g, i and b, d, f, h, j (which are real numbers), and called 5-order sequences.

In this paper, we consider two sequences of t-order $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by $\alpha_0 = a_1, \ \beta_0 = a_2, \ \alpha_1 = a_3$, $\beta_1 = a_4, ..., \alpha_{t-1} = a_{2t-1}, \ \beta_t = a_{2t}$

$$\alpha_{n+t} = \sum_{i=0}^{t-1} \beta_{n+i}, \ \beta_{n+t} = \sum_{i=0}^{t-1} \alpha_{n+i}, \ n \geq 0, \qquad (2)$$

where $a_1, \ a_2, \ldots, a_{2t-1}, \ a_{2t}$ are fixed real numbers. Furthermore, some interesting properties of these sequences are given.

Taking t=2 in (2), the sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ in (1) are obtained:

Table 1. The first nine terms of the sequences of 2-order are shown in table below

n	α_n	β_n
0	a	b
1	c	d
2	b+d	a+c
3	a+c+d	b+c+d
4	a+b+2c+d	a+b+c+2d
5	a+2b+2c+3d	2a+b+3c+2d
6	3a+2b+4c+4d	2a+3b+4c+4d
7	4a+4b+7c+6d	4a+4b+6c+7d
8	6a+7b+10c+11d	7a+6b+11c+10d

2. SOME PROPERTIES RELATED TO THE SEQUENCES OF t-ORDER

In this section, we will give the sums of terms of the sequences of t-order and some interesting results.

Theorem1. For every integer $n \geq 0$ and $0 \leq k \leq t$,

$$\alpha_{(t+1)n+k} + \beta_k = \beta_{(t+1)n+k} + \alpha_k. \qquad (3)$$

Proof. Since $\alpha_k + \beta_k = \beta_k + \alpha_k$, the given statement is clearly true when n=0.

Assume that the result is true for some integer $n \geq 1$. From (2) and induction hypothesis, then

$$\alpha_{(t+1)(n+1)+k}+\beta_k=\sum_{i=1}^{t}\beta_{(t+1)n+k+i}+\beta_k$$
$$=\alpha_{(t+1)n+k+t}+\beta_{k+t}-\alpha_{k+t}+\alpha_{(t+1)n+k+t-1}+\beta_{k+t-1}$$
$$-\alpha_{k+t-1}+\ldots+\alpha_{(t+1)n+k+1}+\beta_{k+1}-\alpha_{k+1}+\beta_k$$
$$=\sum_{i=1}^{t}\alpha_{(t+1)n+k+i}+\sum_{i=1}^{t}\beta_{k+i}-\sum_{i=1}^{t}\alpha_{k+i}+\beta_k$$
$$=\beta_{(t+1)n+k+(t+1)}+\alpha_{k+t+1}-\beta_{k+t+1}+\beta_k$$
$$=\beta_{(t+1)(n+1)+k}+\alpha_k-\beta_k+\beta_k$$
$$=\beta_{(t+1)(n+1)+k}+\alpha_k.$$

So the statement is true for n+1. Thus it is true for every positive integer n. □

For example, taking t=3 in (3), we write

$$\alpha_{4n}+\beta_0=\beta_{4n}+\alpha_0,$$
$$\alpha_{4n+1}+\beta_1=\beta_{4n+1}+\alpha_1,$$
$$\alpha_{4n+2}+\beta_2=\beta_{4n+2}+\alpha_2,$$
$$\alpha_{4n+3}+\beta_3=\beta_{4n+3}+\alpha_3.$$

Theorem2. For every integer $n\geq 1$ and $0\leq k\leq t-1$,

$$\alpha_{tn+k}=\sum_{i=0}^{tn+k-1}\beta_i-\sum_{i=1}^{n-1}\alpha_{ti+k}-\sum_{i=0}^{k-1}\beta_i,$$
$$\beta_{tn+k}=\sum_{i=0}^{tn+k-1}\alpha_i-\sum_{i=1}^{n-1}\beta_{ti+k}-\sum_{i=0}^{k-1}\alpha_i.$$

Proof. The proof is obtained by induction method on n.

Theorem3. For every integer $n\geq 0$ and $t+1\leq k\leq 2t+1$,

$$\alpha_{(t+1)n+k}=\sum_{i=k-t}^{(t+1)n+k-1}\beta_i-\sum_{i=k-t}^{(t+1)n+k-t-1}\alpha_i,$$
$$\beta_{(t+1)n+k}=\sum_{i=k-t}^{(t+1)n+k-1}\alpha_i-\sum_{i=k-t}^{(t+1)n+k-t-1}\beta_i.$$

Proof. For $n=0$, by (2), we have

$$\sum_{i=k-t}^{k-1}\beta_i-\sum_{i=k-t}^{k-t-1}\alpha_i=\beta_{k-t}+\beta_{k-t+1}+\ldots+\beta_{k-1}=\alpha_k$$

Thus the result is true for $n=0$.

Assume that the result is true for some integer $n\geq 1$. From (2), then

$$\sum_{\mathrm{i=k-t}}^{(t+1)(n+1)+k-1}\beta_i-\sum_{\mathrm{i=k-t}}^{(t+1)(n+1)+k-t-1}\alpha_i=\sum_{\mathrm{i=k-t}}^{(t+1)n+t+k}\beta_i-\sum_{\mathrm{i=k-t}}^{(t+1)n+k}\alpha_i$$
$$=\beta_{(t+1)n+t+k}+\ldots+\beta_{(t+1)n+k}+\sum_{\mathrm{i=k-t}}^{(t+1)n+k-1}\beta_i-\sum_{\mathrm{i=k-t}}^{(t+1)n+k-t-1}\alpha_i$$
$$-\alpha_{(t+1)n+k}-\ldots-\alpha_{(t+1)n+k-t}$$
$$=\sum_{\mathrm{i=k-t}}^{(t+1)n+k-1}\beta_i-\sum_{\mathrm{i=k-t}}^{(t+1)n+k-t-1}\alpha_i+\beta_{(t+1)n+k}+\alpha_{(t+1)(n+1)+k}$$
$$-\beta_{(t+1)n+k}-\alpha_{(t+1)n+k}.$$

From induction hypothesis, then

$$\sum_{\mathrm{i=k-t}}^{(t+1)(n+1)+k-1}\beta_i-\sum_{\mathrm{i=k-t}}^{(t+1)(n+1)+k-t-1}\alpha_i$$
$$=\beta_{(t+1)n+k}+\alpha_{(t+1)(n+1)+k}-\beta_{(t+1)n+k}$$
$$=\alpha_{(t+1)(n+1)+k}.$$

Hence the result is true for all integers $n\geq 0$.

□

We express the terms of the sequences of t-order $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$, when $n\geq 0$, as follows:

$$\alpha_n=a_1\gamma_n^1+a_2\gamma_n^2+\ldots+a_{2t-1}\gamma_n^{2t-1}+a_{2t}\gamma_n^{2t}, \quad (4)$$

$$\beta_n=a_1\delta_n^1+a_2\delta_n^2+\ldots+a_{2t-1}\delta_n^{2t-1}+a_{2t}\delta_n^{2t}. \quad (5)$$

Thus, the sequences $\{\gamma_i^j\}_{i=0}^{\infty}$ and $\{\delta_i^j\}_{i=0}^{\infty}$ $(1\leq j\leq 2t)$ are obtained.

Now, we will show how these sequences are related to each other.

Theorem4. For every integer $n\geq 0$ and $1\leq i\leq t$,

$$\delta_n^{2i-1}=\gamma_n^{2i},\ \delta_n^{2i}=\gamma_n^{2i-1}. \quad (6)$$

Proof. For i=1, we prove $\delta_n^1=\gamma_n^2$ and $\delta_n^2=\gamma_n^1$.

We shall apply induction method on n.

For n=0, since $\delta_0^1=0=\gamma_0^2, \delta_0^2=1=\gamma_0^1$, the result is true for n=0.

Assume that the statement is true for all integers less than or equal to some integer $n\geq 1$. From (2) and induction hypothesis, then

$$\delta_{n+1}^1=\gamma_n^1+\gamma_{n-1}^1+\ldots+\gamma_{n-t+1}^1$$
$$=\delta_n^2+\delta_{n-1}^2+\ldots+\delta_{n-t+1}^2=\gamma_{n+1}^2,$$

and

$$\delta_{n+1}^2=\gamma_n^2+\gamma_{n-1}^2+\ldots+\gamma_{n-t+1}^2$$
$$=\delta_n^1+\delta_{n-1}^1+\ldots+\delta_{n-t+1}^1=\gamma_{n+1}^1.$$

Hence the desired statement is true for all integers $n \geq 0$.

Similarly, for $2 \leq i \leq t$, the proof is obtained.

□

Theorem 5. For every integer $n \geq 0$ and $2 \leq i \leq t$,

$$\gamma_{n+1}^{1} + \delta_{n+1}^{1} = \gamma_{n}^{2t-1} + \delta_{n}^{2t-1}, \qquad (8)$$

$$\gamma_{n+1}^{2i-1} + \delta_{n+1}^{2i-1} = \left(\gamma_{n}^{2t-1} + \delta_{n}^{2t-1}\right) + \left(\gamma_{n}^{2i-3} + \delta_{n}^{2i-3}\right). \qquad (9)$$

Proof. The proof is obtained by induction method on n.

Let Ψ be the integer function defined for every $k \geq 0$ by

$$\Psi\left((t+1)k + r\right) = \begin{cases} -1, & if\ r = t, \\ 1, & if\ r = \left\lceil \frac{i}{2} \right\rceil - 1, \\ 0, & otherwise, \end{cases} \qquad (10)$$

where $1 \leq i \leq 2t$.

Obviously, taking $n = (t+1)k + r$ in (10), we write

$$\Psi(n+1) = -\Psi(n) - \Psi(n-1) - \ldots - \Psi(n-t+1).$$

Now, we will give the some relations related to the sequences $\{\gamma_i^j\}_{i=0}^{\infty}$, $\{\delta_i^j\}_{i=0}^{\infty}$ and function $\Psi(n)$.

Theorem 6. For every integer $n \geq 0$ and $1 \leq i \leq 2t$,

$$\gamma_{n}^{i} = \delta_{n}^{i} - (-1)^{i}\Psi(n) \qquad (11)$$

Proof. Using the definition of the function Ψ, the proof is easily obtained by induction on n.

For example, for every integer $n \geq 0$, taking t=3 and i=3 in Theorem 6, we obtain

$$\gamma_{n}^{3} = \delta_{n}^{3} + \Psi(n),$$

where Ψ is the integer function defined for every $k \geq 0$ as follows:

r	$\Psi(4k+r)$
0	0
1	1
2	0
3	-1

Theorem 7. For every integer $n \geq 0$ and $1 \leq j \leq 2t$,

$$\gamma_{n+t}^{j} = \sum_{i=0}^{t-1} \gamma_{n+i}^{j} - (-1)^{j}\Psi(n+t),$$

$$\delta_{n+t}^{j} = \sum_{i=0}^{t-1} \delta_{n+i}^{j} + (-1)^{j}\Psi(n+t).$$

Proof. To prove this, we shall apply induction method on n.

Using (2) and (11), for n=0, we get

$$\begin{aligned} &\sum_{i=0}^{t-1} \gamma_{i}^{j} - (-1)^{j}\Psi(t) \\ &= \gamma_{0}^{j} + \gamma_{1}^{j} + \ldots + \gamma_{t-1}^{j} - (-1)^{j}\Psi(t) \\ &= \delta_{t}^{j} - (-1)^{j}\Psi(t) = \gamma_{t}^{j} \end{aligned}$$

Thus the result is true for $n = 0$.

Assume that the assertion is true for some integer $n \geq 2$. Using (2), (11) and induction hypothesis, then

$$\begin{aligned} &\sum_{i=0}^{t-1} \gamma_{n+1+i}^{j} - (-1)^{j}\Psi(n+1+t) \\ &= \gamma_{n+1}^{j} + \gamma_{n+2}^{j} + \ldots + \gamma_{n+t}^{j} - (-1)^{j}\Psi(n+1+t) \\ &= \delta_{n+t+1}^{j} - (-1)^{j}\Psi(n+1+t) = \gamma_{n+t+1}^{j}. \end{aligned}$$

Hence the result is true for all integers $n \geq 0$.

Similarly, the proof of the other result is obtained.

□

From (4) and (5), we write

$$\begin{aligned} G_n &= \alpha_n + \beta_n \\ &= a_1 G_n^1 + a_2 G_n^2 + \ldots + a_t G_n^t + \ldots + a_{2t} G_n^{2t}, \end{aligned} \qquad (12)$$

and

$$\begin{aligned} H_n &= \alpha_n - \beta_n \\ &= a_1 H_n^1 + a_2 H_n^2 + \ldots + a_t H_n^t + \ldots + a_{2t} H_n^{2t}, \end{aligned} \qquad (13)$$

where $G_n^i = \gamma_n^i + \delta_n^i$, $H_n^i = \gamma_n^i - \delta_n^i$, $1 \leq i \leq 2t.$

Now, we define the integer function θ for every $k \geq 0$ as follows:

$$\theta(n) = \theta((t+1)k + r) = \begin{cases} -1, & if\ r = t, \\ 1, & if\ r = i-1, \\ 0, & otherwise, \end{cases}$$

where $1 \leq i \leq t$.

Theorem8. For every integer $n \geq 0$ and $1 \leq i \leq t$,

$$G_n^{2i-1} = G_n^{2i} = \sum_{k=0}^{i-1} G_{n-k}^{1}, \qquad (15)$$

$$H_n^{2i-1} = -H_n^{2i} = \theta(n). \qquad (16)$$

Proof. Firstly, we prove equality in (15). From (9) and (8), then

$$G_n^{2i-1} = G_{n-1}^{2t-1} + G_{n-1}^{2i-3}$$
$$= G_{n-1}^{2t-1} + G_{n-2}^{2t-1} + G_{n-2}^{2i-5}$$
$$= \ldots = G_{n-1}^{2t-1} + G_{n-2}^{2t-1} + \ldots + G_{n-i}^{2t-1}$$
$$= G_n^1 + G_{n-1}^1 + \ldots + G_{n-i+1}^1 = \sum_{k=0}^{i-1} G_{n-k}^1. \qquad (17)$$

Adding $\gamma_n^{2i} = \delta_n^{2i-1}$ and $\gamma_n^{2i-1} = \delta_n^{2i}$, we have

$$G_n^{2i-1} = G_n^{2i}. \qquad (18)$$

Thus,by (17) and (18), the claimed result is obtained.

Secondly, we prove equality in (16).

By (2) and (13), we have

$$H_{(t+1)n+k+t+1} = \alpha_{(t+1)n+k+t+1} - \beta_{(t+1)n+k+t+1}$$
$$= \beta_{(t+1)n+k+t} - \alpha_{(t+1)n+k+t} + \beta_{(t+1)n+k+t-1}$$
$$- \alpha_{(t+1)n+k+t-1} + \ldots + \beta_{(t+1)n+k+1} - \alpha_{(t+1)n+k+1}$$

Using (2) and (3), for $0 \le k \le t$, we write

$$H_{(t+1)n+k+t+1}$$
$$= \beta_{k+t} - \alpha_{k+t} + \beta_{k+t-1} - \alpha_{k+t-1} + \ldots + \beta_{k+1} - \alpha_{k+1}$$
$$= \alpha_{k+t+1} - \beta_{k+t+1} = \alpha_k - \beta_k = H_k.$$

From definition of H_k in (13) and (2), we get

for $k=0$, $H_0 = \alpha_0 - \beta_0$,

for $k=1$,

$$H_1 = 0(\alpha_0 - \beta_0) + 1(\alpha_1 - \beta_1),$$

…

for $k = t-1$,

$$H_{t-1} = 0(\alpha_0 - \beta_0) + \ldots + 0(\alpha_{t-2} - \beta_{t-2})$$
$$+ 1(\alpha_{t-1} - \beta_{t-1}),$$

for $k = t$,

$$H_t = -(\alpha_0 - \beta_0) - (\alpha_1 - \beta_1) - \ldots - (\alpha_{t-1} - \beta_{t-1}).$$

Since

$$H_n = a_1 H_n^1 + a_2 H_n^2 + \ldots + a_t H_n^t + \ldots + a_{2t} H_n^{2t},$$

the claimed result $H_n^{2i-1} = -H_n^{2i}$ is obtained..

Using $n = (t+1)k + r$ and the integer function θ, the desired result is proved.

□

By (15) and (16), we write

$$\gamma_n^{2i-1} = \frac{G_n^{2i-1} + H_n^{2i-1}}{2} = \frac{\sum_{k=0}^{i-1} G_{n-k}^1 + \theta(n)}{2},$$
$$\gamma_n^{2i} = \frac{G_n^{2i} + H_n^{2i}}{2} = \frac{\sum_{k=0}^{i-1} G_{n-k}^1 - \theta(n)}{2}, \qquad (19)$$

and

$$\delta_n^{2i-1} = \frac{G_n^{2i-1} - H_n^{2i-1}}{2} = \frac{\sum_{k=0}^{i-1} G_{n-k}^1 - \theta(n)}{2},$$
$$\delta_n^{2i} = \frac{G_n^{2i} - H_n^{2i}}{2} = \frac{\sum_{k=0}^{i-1} G_{n-k}^1 + \theta(n)}{2}.$$

Hence, from (4) and (19), then

$$\alpha_n = a_1\gamma_n^1 + a_2\gamma_n^2 + \ldots + a_{2t-1}\gamma_n^{2t-1} + a_{2t}\gamma_n^{2t}$$
$$= a_1 \frac{G_n^1 + \theta(n)}{2} + a_2 \frac{G_n^1 - \theta(n)}{2}$$
$$+ \ldots + a_{2t} \frac{\sum_{k=0}^{t-1} G_{n-k}^1 - \theta(n)}{2}$$
$$= \frac{1}{2}(a_1 G_n^1 + \ldots + a_{2t} \sum_{k=0}^{t-1} G_{n-k}^1) \qquad (20)$$
$$+ \frac{\theta(n)}{2}(a_1 - a_2 + \ldots - a_{2t})$$
$$= \frac{1}{2}\sum_{j=0}^{t-1}\sum_{i=1}^{2(t-j)} a_{i+2j} G_{n-j}^1 + \frac{\theta(n)}{2}\sum_{i=1}^{t}(a_{2i-1} + a_{2i}).$$

Similarly, the result for β_n is obtained.

For example, taking t=2 in (20), we write

$$\alpha_n = \frac{1}{2}\Big((G_n^1 + \theta(n))a_1 + (G_n^1 - \theta(n))a_2$$
$$+ (G_n^1 + G_{n-1}^1 + \theta(n))a_3 + (G_n^1 + G_{n-1}^1 - \theta(n))a_4\Big).$$

From (12), then

$$\alpha_n = \frac{1}{2}\Big((\gamma_n^1 + \delta_n^1 + \theta(n))a_1 + (\gamma_n^1 + \delta_n^1 - \theta(n))a_2$$
$$+ (\gamma_n^1 + \delta_n^1 + \gamma_{n-1}^1 + \delta_{n-1}^1 + \theta(n))a_3$$
$$+ (\gamma_n^1 + \delta_n^1 + \gamma_{n-1}^1 + \delta_{n-1}^1 - \theta(n))a_4\Big).$$

Since $\gamma_n^1 + \delta_n^1 = F_{n-1}$ in [1], then

$$\alpha_n = \frac{1}{2}\Big((F_{n-1} + \theta(n))a_1 + (F_{n-1} - \theta(n))a_2$$
$$+ (F_n + \theta(n))a_3 + (F_n - \theta(n))a_4\Big),$$

where F_n is n th Fibonacci number.

3. CONCLUSION

In this study, the sequences of t-order are defined and some properties are given. In future, we define the sequences of t-order under different schemes and the results are obtained for these schemes.

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