Coupled Fixed Point Theorems Under Nonlinear Contractive Conditions in G-Metric spaces

Hans Raj
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science & Technology,
Murthal, Sonepat, Haryana, India

Nawneet Hooda
Department of Mathematics,
Deenbandhu Chhotu Ram University of Science & Technology,

ABSTRACT

The aim of this paper is to prove a number of coupled fixed point theorems under φ-contractions for a mapping $F : X \times X \to X$ in G-metric spaces. The given result is conversion of the result of Erdal Karapinar et al. [3] into G-metric space with coupled fixed point theorem.

General Terms:

54A05

Keywords:

G-metric space, Coupled fixed point, Mixed monotone property

1. INTRODUCTION

The Banach contraction principle is the most powerful tool in the history of fixed point theory. Boyd and Wong [2] extended the Banach contraction principle to the nonlinear contraction mappings. The notion of coupled fixed point was intiated by Gnana Bhaskar and Lakshmikantham [1] in 2006. After this many authors worked on coupled fixed point theorems. The notion of G-metric space is given by Mustafa and Sims [6] as a generalization of metric spaces in 2006. Based on the concept of G-metric spaces, Mustafa et al. [7, 4, 5] proved several fixed point theorems for mappings satisfying different contractive conditions.

2. PRELIMINARIES

DEFINITION 2.1 SEE [6]. *Let X be a nonempty set, and let $G : X \times X \times X \to R^+$, be a function satisfying:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$

(G2) $0 < G(x, x, y)$, *for all* $x, y \in X$*; with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$,
for all $x, y, z \in X$ *with* $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$
(*symmetry in all three variables*), *and*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all*
$x, y, z, a \in X$, (*rectangle inequality*),

then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

DEFINITION 2.2 SEE [6]. *A G-metric space (X, G) is symmetric if*

(G6) $G(x, y, y) = G(x, x, y)$, *for all* $x, y \in X$.

DEFINITION 2.3 SEE [1]. *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$F(x, y) = x \quad \textit{and} \quad F(y, x) = y.$$

DEFINITION 2.4 SEE [1]. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping.*
F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y), \quad \textit{for } x_1, x_2 \in X$$

and

$$y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_2) \leq F(x, y_1), \quad \textit{for } y_1, y_2 \in X.$$

By following Matkowski [7], we let Φ be the set of all nondecreasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to+\infty} \varphi n(t) = 0$ for all $t > 0$. Then, it is an easy matter to show that

(1) $\phi(t) < 0$ *for all* $t > 0$,

(2) $\phi(0) = 0$.

In this paper, some coupled fixed point theorems are proved for a mapping $F : X \times X \to X$ satisfying a contractive condition based on some $\varphi \in \Phi$.

3. MAIN RESULTS

THEOREM 3.1. *Let $(X, \leq)$ be a partially ordered set and (X, G) a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that F has the mixed monotone property.*

Assume that there exists $\varphi \in \Phi$ such that

$$G(F(x,y),F(u,v),F(u,v)) \leq \varphi[\max(G(x,u,u),G(y,v,v))] \quad (1)$$

for all $x,y,u,v \in X$ with $x \geq u$ and $y \leq v$.
If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$, then F has a coupled fixed point.

PROOF. Suppose $x_0, y_0 \in X$ are such that $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$. Define

$$x_1 = F(x_0,y_0) \quad \text{and} \quad y_1 = F(y_0,x_0). \quad (2)$$

Then, $x_0 \leq x_1$, $y_0 \geq y_1$. Again, define $x_2 = F(x_1,y_1)$ and $y_2 = F(y_1,x_1)$. Since F has the mixed monotone property, we have $x_0 \leq x_1 \leq x_2$ and $y_2 \leq y_1 \leq y_0$.
Continuing like this, we can construct two sequences x_n and y_n in X such that

$$\left.\begin{aligned} &x_n = F(x_{n-1},y_{n-1}) \leq x_{n+1} = F(x_n,y_n) \text{ and} \\ &y_{n+1} = F(y_n,x_n) \leq y_n = F(y_{n-1},x_{n-1}). \end{aligned}\right\} \quad (3)$$

if, for some integer n, we have

$$(x_{n+1},y_{n+1}) = (x_n,y_n),$$

then

$$x_n = F(x_n,y_n) \quad \text{and} \quad y_n = F(y_n,x_n)$$

that is, (x_n,y_n) is a coupled fixed point of F. Thus, we suppose that $(x_{n+1},y_{n+1}) \neq (x_n,y_n)$ for all $n \in N$; that is, we assume that either $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$. For any $n \in N$, we have

$$\left.\begin{aligned} &G(x_{n+1},x_n,x_n) \\ &\quad = G(F(x_n,y_n),F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1})) \\ &\quad \leq \varphi(\max(G(x_n,x_{n-1},x_{n-1}),G(y_n,y_{n-1},y_{n-1})) \\ &G(y_n,y_{n+1},y_{n+1}) \\ &\quad = G(F(y_{n-1},x_{n-1}),F(y_n,x_n),F(y_n,x_n)) \\ &\quad \leq \varphi(\max(G(y_{n-1},y_n,y_n),G(x_{n-1},x_n,x_n)) \end{aligned}\right\} \quad (4)$$

From eq. (4) we get that

$$\max(G(x_{n+1},x_n,x_n),G(y_n,y_{n+1},y_{n+1})) \leq \varphi(\max(G(x_n,x_{n-1},x_{n-1}),G(y_{n-1},y_n,y_n)) \quad (5)$$

By continuing the process of eq. (5) we get

$$\left.\begin{aligned} &\max(G(x_{n+1},x_n,x_n),G(y_n,y_{n+1},y_{n+1})) \\ &\leq \varphi(\max(G(x_n,x_{n-1},x_{n-1}),G(y_{n-1},y_n,y_n)) \\ &\leq \varphi^2(\max(G(x_{n-1},x_{n-2},x_{n-2}),G(y_{n-2},y_{n-1},y_{n-1})) \\ &\vdots \\ &\leq \varphi^n(\max(G(x_1,x_0,x_0),G(y_0,y_1,y_1)) \end{aligned}\right\} \quad (6)$$

Now, we will show that x_n and y_n are Cauchy sequences in X. Let $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \varphi^n(\max(G(x_1,x_0,x_0),G(y_0,y_1,y_1)) = 0 \quad (7)$$

and $\varepsilon > \varphi(\varepsilon)$, there exist $n_0 \in N$ such that

$$\varphi^n(\max(G(x_1,x_0,x_0),G(y_0,y_1,y_1)) < \varepsilon - \varphi(\varepsilon) \quad \text{for all } n = n_0 \quad (8)$$

This implies that

$$\max(G(x_{n+1},x_n,x_n),G(y_n,y_{n+1},y_{n+1})) < \varepsilon - \varphi(\varepsilon) \quad \text{for all } n = n_0 \quad (9)$$

For $m,n \in N$, we will prove by induction on m that

$$\max(G(x_n,x_m,x_m),G(y_n,y_m,y_m)) < \varepsilon \quad \text{for all } m \geq n \geq n_0 \quad (10)$$

Since $\varepsilon - \varphi(\varepsilon) < \varepsilon$, then by using (9) we get that (10) holds when $m = n+1$. Now suppose that (10) holds for $m = k$. For $m = k+1$, we have

$$\begin{aligned} &G(x_n,x_{k+1},x_{k+1}) \\ &\leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{k+1},x_{k+1}) \\ &\leq \varepsilon - \varphi(\varepsilon) + G(F(x_n,y_n),F(x_k,y_k),F(x_k,y_k)) \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi(\max(G(x_n,x_k,x_k),G(y_n,y_k,y_k)) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon). \end{aligned} \quad (11)$$

Similarly, we show that

$$G(y_n,y_{k+1},y_{k+1}) < \varepsilon\,. \quad (12)$$

Hence, we have

$$\max(G(x_n,x_{k+1},x_{k+1}),G(y_n,y_{k+1},y_{k+1})) < \varepsilon \quad (13)$$

Thus, (10) holds for all $m \geq n \geq n_0$. Hence, x_n and y_n are Cauchy sequences in X.
Since X is a complete G-metric space, there exist x and $y \in X$ such that x_n and y_n converge to x and y respectively. Finally, we show that (x,y) is a coupled fixed point of F. Since F is continuous and $(x_n,y_n) \to (x,y)$. we have

$$x_{n+1} = F(x_n,y_n) \to F(x,y).$$

By the uniqueness of limit, we get that $x = F(x,y)$. Similarly, we show that $y = F(y,x)$.
So, (x,y) is a coupled fixed point of F. □

By taking $\varphi(t) = kt$, where $k \in (0,1]$, in Theorem 3.1, we have the following.

COROLLARY 3.2. *Let $(X,\leq)$ be a partially ordered set and (X,G) a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that*

$$(F(x,y),F(u,v),F(u,v)) \leq k(\max(G(x,u,u),G(y,v,v)) \quad (14)$$

for all $x,y,u,v \in X$ with $x \geq u$ and $y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0,y_0)$ and $y_0 = F(y_0,x_0)$, then F has a coupled fixed point.

As a consequence of Corollary 3.2, we have the following.

COROLLARY 3.3. *Let $(X,\leq)$ be a partially ordered set and (X,G) a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exists $a_1, a_2 \in [0,1)$ such that*

$$(F(x,y),F(u,v),F(u,v)) \leq a_1(G(x,u,u) + a_2G(y,v,v)) \quad (15)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \geq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

THEOREM 3.4. *Let $(X, \leq)$ be a partially ordered set and (X, G) a complete G-metric space. Let $F : X \times X \to X$ be a mapping having mixed monotone property. Assume that there exists $\varphi \in \Phi$ such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq \varphi[\max(G(x,u,u), G(y,v,v))] \quad (16)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Assume also that X has the following properties:

(i) if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in N$,
(ii) if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

PROOF. By following the same process in Theorem 3.1, we construct two Cauchy sequences x_n and y_n in X with

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \text{ and } y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots \quad (17)$$

such that $x_n \to x \in X$ and $y_n \to y \in X$. By the hypotheses on X, we have $x_n \leq x$ and $y_n \geq y$ for all $n \in N$. From (16), we have

$$\begin{aligned} &G(F(x,y), x_{n+1}, x_{n+1}) \\ &\quad = G(F(x,y), F(x_n, y_n), F(x_n, y_n)) \\ &\quad \leq \varphi(\max(G(x, x_n, x_n), G(y, y_n, y_n)) \\ &G(y_{n+1}, F(y,x), F(y,x)) \\ &\quad = G(F(y_n, x_n), F(y,x), F(y,x)) \\ &\quad = \varphi(\max(G(y_n, y, y), G(x_n, x, x)) \end{aligned} \quad (18)$$

From (18) we get,

$$\begin{aligned} &\max[G(F(x,y), x_{n+1}, x_{n+1}), G(y_{n+1}, F(y,x), F(y,x))] \\ &\quad \leq \varphi(\max[G(x, x_n, x_n), G(y, y_n, y_n)), \\ &\qquad\qquad G(y_n, y, y), G(x_n, x, x))]. \end{aligned} \quad (19)$$

Letting $n \to +\infty$ in (19), it follows that $x = F(x, y)$ and $y = F(y, x)$. Hence (x, y) is a coupled fixed point of F. □

By taking $\varphi(t) = kt$, where $k \in (0, 1]$, in Theorem 3.4, we have the following result.

COROLLARY 3.5. *Let $(X, \leq)$ be a partially ordered set and (X, G) a complete G-metric space. Let $F : X \times X \to X$ be a mapping having mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq k \max(G(x,u,u), G(y,v,v)) \quad (20)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.
Assume also that X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in N$,*
(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in N$.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

As a consequence of Corollary 3.5, we have the following.

COROLLARY 3.6. *Let $(X, \leq)$ be a partially ordered set and (X, G) a complete G-metric space. Let $F : X \times X \to X$ be a mapping having mixed monotone property. Assume that there exists $a_1, a_2 \in [0, 1)$ such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq a_1(G(x,u,u) + a_2 G(y,v,v)) \quad (21)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.
Assume also that X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n \in N$,*
(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$ for all $n \in N$.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point.

4. REFERENCES

[1] T.G. Bhaskar and V. Lakshmikantham. Fixed point theory in partially ordered metric spaces and applications. *Nonlinear Anal.*, 65.

[2] D.W. Boyd and S.W. Wong. On nonlinear contractions. *Proc. Am. Math. Soc.*, 20.

[3] W. Shatanawi Erdal Karapinar and Z. Mustafa. Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces. *Journal of Applied Mathematics*.

[4] Z. Mustafa. A new structure for generalized metric spaces with applications to fixed point theory.

[5] Z. Mustafa and B. Sims. Fixed point theorems for contractive mappings in complete G-metric spaces. *Fixed Point Theory Appl.*, 10.

[6] Z. Mustafa and B. Sims. A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.*, 7(2):289–297, 2006.

[7] H. Obiedat Z. Mustafa and F. Awawdeh. Some fixed point theorem for mapping on complete metric spaces. *Fixed Point Theory Appl.*, 12.