Coupled Fixed Point Theorems Under Nonlinear Contractive Conditions in G-Metric spaces

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ABSTRACT

The aim of this paper is to prove a number of coupled fixed point theorems under \( \phi \)-contractions for a mapping \( F : X \times X \to X \) in G-metric spaces. The given result is conversion of the result of Erdal Karapinar et al. [3] into G-metric space with coupled fixed point theorem.

General Terms:
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Keywords:
G-metric space, Coupled fixed point, Mixed monotone property

1. INTRODUCTION

The Banach contraction principle is the most powerful tool in the history of fixed point theory. Boyd and Wong [2] extended the Banach contraction principle to the nonlinear contraction mappings. The notion of coupled fixed point was initiated by Gnana Bhaskar and Lakshmikantham [1] in 2006. After this many authors worked on coupled fixed point theorems. The notion of G-metric space is given by Mustafa and Sims [6] as a generalization of metric spaces in 2006. Based on the concept of G-metric spaces, Mustafa et al. [2] [4] [5] proved several fixed point theorems for mappings satisfying different contractive conditions.

2. PRELIMINARIES

\textbf{Definition 2.1} [6]. Let \( X \) be a nonempty set, and let \( G : X \times X \times X \to R^+ \) be a function satisfying:

- (G1) \( G(x, y, z) = 0 \) if \( x = y = z \)
- (G2) \( 0 < G(x, x, y), \) for all \( x, y \in X; \) with \( x \neq y, \)
- (G3) \( G(x, x, y) \leq G(x, y, z), \) for all \( x, y, z \in X \) with \( z \neq y, \)
- (G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \) (symmetry in all three variables), and
- (G5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z), \) for all \( x, y, z, a \in X, \) (rectangle inequality),

then the function \( G \) is called a generalized metric, or, more specifically a G-metric on \( X, \) and the pair \( (X, G) \) is a G-metric space.

\textbf{Definition 2.2} [6]. A G-metric space \((X, G)\) is symmetric if

\( (G6) \ G(x, y, y) = G(x, x, y), \) for all \( x, y \in X. \)

\textbf{Definition 2.3} [1]. An element \((x, y) \in X \times X\) is said to be a coupled fixed point of the mapping \( F : X \times X \to X \) if

\[ F(x, y) = x \quad \text{and} \quad F(y, x) = y. \]

\textbf{Definition 2.4} [1]. Let \((X, \leq)\) be a partially ordered set and \( F : X \times X \to X \) be a mapping. \( F \) is said to have the mixed monotone property if \( F(x, y) \) is monotone non-decreasing in \( x \) and is monotone non-increasing in \( y, \) that is, for any \( x, y \in X, \)

\[ x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for} \quad x_1, x_2 \in X \]

and

\[ y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \quad \text{for} \quad y_1, y_2 \in X. \]

By following Matkowski [7], we let \( \Phi \) be the set of all nondecreasing functions \( \phi : [0, +\infty) \to [0, +\infty) \) such that \( \lim_{t \to +\infty} \phi(t) = 0 \) for all \( t > 0. \) Then, it is an easy matter to show that

\begin{align*}
(1) & \quad \phi(t) < 0 \text{ for all } t > 0, \\
(2) & \quad \phi(0) = 0.
\end{align*}

In this paper, some coupled fixed point theorems are proved for a mapping \( F : X \times X \to X \) satisfying a contractive condition based on some \( \phi \in \Phi. \)

3. MAIN RESULTS

\textbf{Theorem 3.1}. Let \((X, \leq)\) be a partially ordered set and \((X, G)\) a complete G-metric space. Let \( F : X \times X \to X \) be a continuous mapping such that \( F \) has the mixed monotone property.
Assume that there exists $\varphi \in \Phi$ such that
\[
G(F(x,y), F(u,v), F(v,u)) \\
\leq \varphi(\max(G(x,x,u), G(y,v,v)))
\]
for all $x, y, u, v$ in $X$ with $x \geq u$ and $y \leq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then $F$ has a coupled fixed point.

PROOF. Suppose $x_0, y_0 \in X$ are such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

Define
\[
x_1 = F(x_0, y_0)
\quad \text{and} \quad
y_1 = F(y_0, x_0).
\]
Then, $x_0 \leq x_1$, $y_0 \geq y_1$. Again, define $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Since $F$ has the mixed monotone property, we have $x_0 \leq x_1 \leq x_2$ and $y_0 \geq y_1 \geq y_2$. Continuing like this, we can construct two sequences $x_n$ and $y_n$ in $X$ such that
\[
x_n = F(x_{n-1}, y_{n-1}) \leq x_{n+1} = F(x_n, y_n)
\quad \text{and} \quad
y_n = F(y_{n-1}, x_{n-1}) \leq y_{n+1} = F(y_n, x_n).
\]

If, for some integer $n$, we have
\[
(x_{n+1}, y_{n+1}) = (x_n, y_n),
\]
then
\[
x_n = F(x_n, y_n)
\quad \text{and} \quad
y_n = F(y_n, x_n)
\]
that is, $(x_n, y_n)$ is a coupled fixed point of $F$. Thus, we suppose that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \in N$; that is, we assume that either $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$. For any $n \in N$, we have
\[
\begin{align*}
G(x_{n+1}, x_n, x_n) &= G(F(x_n, y_n), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\
\quad &\leq \varphi(\max(G(x_n, x_n, x_n), G(y_{n-1}, y_{n-1}, y_{n-1}))) \\
G(y_{n+1}, y_n, y_n) &= G(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n)) \\
\quad &\leq \varphi(\max(G(y_{n-1}, y_n, y_n), G(x_{n-1}, x_n, x_n))).
\end{align*}
\]

From eq. (4) we get that
\[
\max(G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n)) \\
\leq \varphi(\max(G(x_{n-1}, x_{n-1}, x_{n-1}), G(y_{n-1}, y_{n-1}, y_{n-1}))).
\]

By continuing the process of eq. (5) we get
\[
\max(G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_n)) \\
\leq \varphi(\max(G(x_{n+1}, x_{n-1}, x_{n-1}), G(y_{n+1}, y_{n-1}, y_{n-1}))) \\
\leq \varphi^2(\max(G(x_{n+1}, x_{n-2}, x_{n-2}), G(y_{n+1}, y_{n-2}, y_{n-1}))) \\
\vdots \\
\leq \varphi^n(\max(G(x_1, x_0, x_0), G(y_0, y_1, y_1))).
\]

Now, we will show that $x_n$ and $y_n$ are Cauchy sequences in $X$. Let $\varepsilon > 0$. Since
\[
\lim_{n \to \infty} \varphi^n(\max(G(x_1, x_0, x_0), G(y_0, y_1, y_1))) = 0
\]
and $\varepsilon > \varphi(\varepsilon)$, there exist $n_0 \in N$ such that
\[
\varphi^n(\max(G(x_1, x_0, x_0), G(y_0, y_1, y_1))) < \varepsilon - \varphi(\varepsilon)
\quad \text{for all } n = n_0.
\]

This implies that
\[
\max(G(x_{n+1}, x_n, x_n), G(y_{n+1}, y_n, y_{n+1})) < \varepsilon - \varphi(\varepsilon)
\quad \text{for all } n = n_0
\]

For $m, n \in N$, we will prove by induction on $m$ that
\[
\max(G(x_m, x_m, x_m), G(y_m, y_m, y_m)) < \varepsilon
\quad \text{for all } m \geq n \geq n_0
\]

Since $\varepsilon - \varphi(\varepsilon) < \varepsilon$, then by using (9) we get that (10) holds when $m = n + 1$. Now suppose that (10) holds for $m = k$. For $m = k + 1$, we have
\[
\max(G(x_{k+1}, x_{k+1}, x_{k+1}), G(y_{k+1}, y_{k+1}, y_{k+1})) < \varepsilon
\]

Similarly, we show that
\[
\max(G(x_{k+1}, x_{k+1}, x_{k+1}), G(y_{k+1}, y_{k+1}, y_{k+1})) < \varepsilon
\]

Hence, we have
\[
\max(G(x_m, x_{m+1}, x_{m+1}), G(y_m, y_{m+1}, y_{m+1})) < \varepsilon
\]

Thus, (10) holds for all $m \geq n \geq n_0$. Hence, $x_n$ and $y_n$ are Cauchy sequences in $X$.

Since $X$ is a complete G-metric space, there exist $x$ and $y \in X$ such that $x_n$ and $y_n$ converge to $x$ and $y$ respectively. Finally, we show that $(x, y)$ is a coupled fixed point of $F$. Since $F$ is continuous and $(x_n, y_n) \to (x, y)$, we have
\[
x_{n+1} = F(x_n, y_n) \to F(x, y).
\]

By the uniqueness of limit, we get that $x = F(x, y)$. Similarly, we show that $y = F(y, x)$.

So, $(x, y)$ is a coupled fixed point of $F$.

By taking $\varphi(t) = kt$, where $k \in (0, 1]$, in Theorem 3.1, we have the following.

**COROLLARY 3.2.** Let $(X, \leq)$ be a partially ordered set and $(X, G)$ a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that $F$ has the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that
\[
(F(x,y), F(u,v), F(v,u)) \\
\leq k(\max(G(x,u,u), G(y,v,v)))
\]
for all $x, y, u, v \in X$ with $x \geq y \geq u$ and $y \leq v \leq u$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then $F$ has a coupled fixed point.

As a consequence of Corollary 3.2, we have the following.

**COROLLARY 3.3.** Let $(X, \leq)$ be a partially ordered set and $(X, G)$ a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping such that $F$ has the mixed monotone property. Assume that there exists $a_1, a_2 \in [0, 1)$ such that
\[
(F(x,y), F(u,v), F(v,u)) \\
\leq a_1(\max(G(x,u,u) + a_2G(y,v,v)))
\]
for all \( x, y, u, v \in X \) with \( x \geq u \) and \( y \geq v \). If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then \( F \) has a fixed pointed.

**Theorem 3.4.** Let \((X, \leq)\) be a partially ordered set and \((X, G)\) a complete G-metric space. Let \( F : X \times X \to X \) be a mapping having mixed monotone property. Assume that there exists \( \varphi \in \Phi \) such that
\[
G(F(x, y), F(u, v), F(u, v)) \\
\leq \varphi(\max(G(x, u, u), G(y, v, v)))
\]
for all \( x, y, u, v \in X \) with \( x \geq u \) and \( y \leq v \).

Assume also that \( X \) has the following properties:

(i) if a nondecreasing sequence \( x_n \to x \), then \( x_n \leq x \) for all \( n \in N \).

(ii) if a nonincreasing sequence \( y_n \to y \), then \( y_n \geq y \) for all \( n \in N \).

If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then \( F \) has a coupled fixed point.

**Proof.** By following the same process in Theorem 3.1, we construct two Cauchy sequences \( x_n \) and \( y_n \) in \( X \) with
\[
x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots
\]
such that \( x_n \to x \in X \) and \( y_n \to y \in X \). By the hypotheses on \( X \), we have \( x_n \leq x \) and \( y_n \geq y \) for all \( n \in N \). From (16), we have
\[
G(F(x, y), F(x_n, x_{n+1}), F(x_n, y_n)) \\
= G(F(x, y), F(x_n, y_n)) \\
\leq \varphi(\max(G(x, x_n, x_n), G(y, y_n, y_n)))
\]
which is Cauchy. Since \( (X, G) \) is a complete G-metric space, \( x_n \to x \) where \( x \) is a coupled fixed point of \( F \).

**Corollary 3.5.** Let \((X, \leq)\) be a partially ordered set and \((X, G)\) a complete G-metric space. Let \( F : X \times X \to X \) be a mapping having mixed monotone property. Assume that there exists \( k \in [0, 1) \) such that
\[
G(F(x, y), F(u, v), F(u, v)) \\
\leq k \max(G(x, u, u), G(y, v, v))
\]
for all \( x, y, u, v \in X \) with \( x \geq u \) and \( y \leq v \).

Assume also that \( X \) has the following properties:

(i) if a nondecreasing sequence \( x_n \to x \), then \( x_n \leq x \) for all \( n \in N \).

(ii) if a nonincreasing sequence \( y_n \to y \), then \( y_n \geq y \) for all \( n \in N \).

If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then \( F \) has a coupled fixed point.

As a consequence of Corollary 3.5, we have the following.

**Corollary 3.6.** Let \((X, \leq)\) be a partially ordered set and \((X, G)\) a complete G-metric space. Let \( F : X \times X \to X \) be a mapping having mixed monotone property. Assume that there exists \( a_1, a_2, \in [0, 1) \) such that
\[
G(F(x, y), F(u, v), F(u, v)) \\
\leq a_1(G(x, u, u) + a_2 G(y, v, v))
\]
for all \( x, y, u, v \in X \) with \( x \geq u \) and \( y \leq v \).

Assume also that \( X \) has the following properties:

(i) if a nondecreasing sequence \( x_n \to x \), then \( x_n \leq x \) for all \( n \in N \).

(ii) if a nonincreasing sequence \( y_n \to y \), then \( y_n \geq y \) for all \( n \in N \).

If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then \( F \) has a coupled fixed point.

**References**


