A New Algorithm for Finding a Minimum Dominating Set of Graphs

P. Pradhan,
Department of Mathematics and Statistics,
Gurukula Kangri University, Haridwar (UK), India

B. S. Rawat,
Department of Mathematics and Statistics,
Gurukula Kangri University, Haridwar (UK), India

ABSTRACT
In the present paper the concept of relative domination power of vertices for finite undirected graphs have been introduced. An algorithm has been developed to obtain a minimum dominating set of a graph. Some results related to domination number γ and other graph theoretic parameters for a tree also obtained.

Keywords
Dominating set, Domination number, Domination power of a vertex, Support of a graph

1. INTRODUCTION
All graphs considered in this paper are undirected and finite. Most of the graph theoretic notations have been borrowed from the book by Harary [1]. A graph $G = (V_G, E_G)$ consist a finite set of vertices V_G together with an edge set E_G such that each edge is an unordered pair of vertices. A subset D of V_G is said to be dominating set of a graph G if every vertex which is not in D is adjacent to at least one vertex in D. The domination number γ of G is the cardinality of smallest dominating set of G. A dominating set D is said to be connected dominating set if induced subgraph $<D>$ is connected [5]. D is called independent dominating set if induced subgraph $<D>$ is null graph. The connected domination number γ_c and the independent domination number γ_i of G are the cardinality of smallest connected dominating set and smallest independent dominating set of G respectively.

A vertex which is adjacent to pendent vertex is called support of G [2]. All vertices of G except pendent vertices are internal vertices. $N(v) = \{u \in V_G \mid uv \in E_G\}$ is called neighbourhood of v in G and the closed neighbourhood of v is denoted by $\overline{N}(v)$ and is defined as $\overline{N}(v) = N(v) \cup \{v\}$. A set of vertices is said to be an independent set of vertices if no two vertices in the set are adjacent and the number of vertices in the largest independent set is called the independence number of a graph [3].

The distance between two vertices u and v of a graph G is denoted by $d(u,v)$ and is defined as the length of shortest path between them.

2. SOME DEFINITIONS

2.1 Definition
A vertex v in G dominates another vertex u in G if $d(u,v) \le 1$. Obviously $d(u,u) = 0 \le 1$, so u dominates itself.

2.2 Definition
A vertex $v \in V_G$ dominates itself and all the vertices adjacent to it, that is v dominates every vertex in its closed neighborhood [4]. The number of vertices in G which are dominated by v is known as domination power of v, it is denoted by $dp(v)$ so $dp(v) = |\overline{N}(v)|$.

2.3 Definition
The relative domination power of a vertex $v \in V_G$ with respect to any vertex $u \in V_G$ is denoted by $dp_u(v)$ and defined as the number of vertices which are dominated by v but not by u. Similarly the relative domination power of a vertex $v \in V_G$ with respect to $u_i, u_j, \dots, u_k$ in V_G is given by $dp_{u_i,u_j,\dots,u_k}(v) =$ Number of vertices dominated by v but not by $u_i, u_j, \dots, u_k$.

2.4 On the Basis of Above Definition 2.3 We Have the Following Observation

2.4.1 Observation
If G is a tree and $d_{v_j}(v_i) = 0$; v_j is an internal vertex and adjacent to vertex v_i, then v_i is a pendent vertex.

2.4.2 Observation
If for a graph G, $dp_{v_j}(v_i) = 0; \forall v_i, v_j \in V_G$, then G is a complete graph.

2.4.3 Observation
If for a graph G, $v_i v_j \in E_G$ such that $dp_{v_j}(v_i) = 1; \forall v_i, v_j \in V_G$ and $|V_G| > 3$, then G is a cycle.

2.4.4 Observation
If for a graph G, $v_i v_j \in E_G$, then $dp_{v_j}(v_i) \le \deg(v_i) - 1$, $\forall v_i, v_j \in V_G$.

2.5 EXAMPLE
In the following graph $dp(v_1) = 3$, $dp(v_2) = 4$, $dp(v_3) = 2$, $dp(v_4) = 2$, $dp(v_5) = 4$, $dp_{v_2}(v_1) = 1$, $dp_{v_5}(v_1) = 1$, $dp_{v_1}(v_2) = 2$, $dp_{v_3}(v_2) = 2$, $dp_{v_4}(v_2) = 2$, $dp_{v_2}(v_3) = 0$, $dp_{v_1 v_3}(v_2) = 1$, $dp_{v_1 v_3 v_4}(v_2) = 0$ etc.

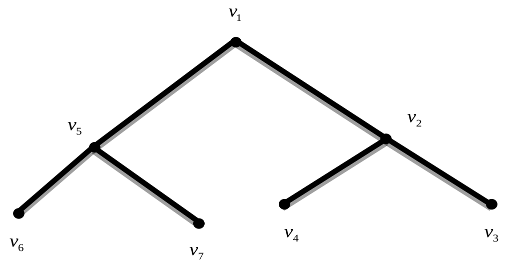

Fig 1: Tree with seven vertices

3. AN ALGORITHM FOR FINDING A MINIMUM DOMINATING SET OF A GRAPH G BY USING RELATIVE DOMINATION POWER OF VERTICES

Case I –When set of supports exist in G

1. Let $G = (V_G, E_G)$ be a graph of order n with t supports.

2. Let $S = \{v_1, v_2, v_3, ..., v_t\}$ be the set of supports such that $dp(v_1) = k_1, dp_{v_1}(v_2) = k_2, dp_{v_1,v_2}(v_3) = k_3 ..., dp_{v_1,v_2,v_3...v_{t-1}}(v_t) = k_t$.

3. If $k_1 + k_2 + k_3 + ... + k_t = n$ and $dp_{v_1,v_2,v_3...v_t}(v_i) = 0; \forall v_i \notin S$ then $D = S$ is a minimum dominating set and domination number $\gamma = |S|$.

4. If $k_1 + k_2 + k_3 + ... + k_t < n,$ then we take a set S_1 such that $S_1 = S \cup \{v_i\}$; where v_i is an internal vertex except support and having maximum relative domination power with respect to $v_1, v_2, v_3, ...v_t$. Let $dp_{v_1,v_2,v_3,...v_t}(v_i) = k_{t+1}$ and $S_1 = \{v_1, v_2, \ v_3, ..., v_t, v_i\}$.

Again if $k_1 + k_2 + k_3 + ... + k_t + k_{t+1} = n$ and $dp_{v_1,v_2,v_3,...v_t,v_i}(v_j) = 0; \forall v_j \notin S_1$. Then $D = S_1$ is a minimum dominating set and domination number $\gamma = |S_1|$.

5. If $k_1 + k_2 + k_3 + ... + k_t + k_{t+1} < n,$ then the process as given in step 4 is continued in similar manner until

$$\sum_{1 \le i \le n} k_i = n \ .$$

Case II –When set of supports does not exist in G

If a graph does not have any support then in order to find a minimum dominating set for a graph; first we select a vertex $v_i \in V_G$ of maximum domination power among all the vertices of G.

Now if $dp(v_i) = n,$ then the set $D = \{v_i\}$ is a minimum dominating set but if $dp(v_i) < n,$ then the process as given above in step 3 and 4 is continued in similar manner until to get a dominating set.

3.1 Example

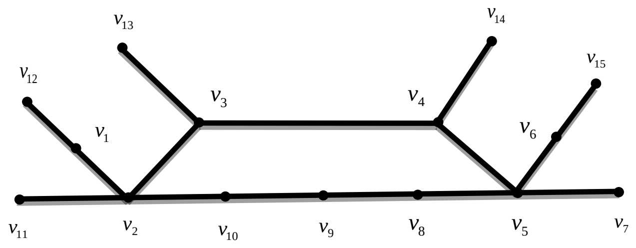

Fig 2: Graph with six supports and fifteen vertices

1. Let us consider the Fig.-2 of the graph $G = (V_G, E_G)$ of order $n = 15$.

2. Let $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the set containing all the supports therefore $|S| = t = 6$ and $dp(v_1) = 3,\ dp_{v_1}(v_2) = 3,\ dp_{v_1,v_2}(v_3) = 2,\ dp_{v_1,v_2,v_3}(v_4) = 2,\ dp_{v_1,v_2,v_3,v_4}(v_5) = 3,\ dp_{v_1,v_2,v_3,v_4,v_5}(v_6) = 1$ that is $k_1 = 3, k_2 = 3, k_3 = 2, k_4 = 2, k_5 = 3, k_6 = 1.$

3. $k_1 + k_2 + k_3 + ... + k_6 = 14 < n$. At this stage we have only three internal vertices except supports v_8, v_9, v_{10} sucht that $dp_{v_1,v_2,v_3,v_4,v_5,v_6}(v_8) = 1,\ dp_{v_1,v_2,v_3,v_4,v_5,v_6}(v_9) = 1,\ dp_{v_1,v_2,v_3,v_4,v_5,v_6}(v_{10}) = 1$ and we see that v_8, v_9 and v_{10} have same relative domination power with respect to $v_1, v_2, v_3, v_4, v_5, v_6$ so we can take anyone vertex out of v_8, v_9 and v_{10}. Suppose we select v_9 then $k_7 = 1$. Hence $S_1 = S \cup \{v_9\} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\}$.

4. Now $k_1 + k_2 + k_3 + ... + k_6 + k_7 = 15 = n$ and $dp_{v_1,v_2,v_3,v_4,v_5,v_6,v_9}(v_i) = 0;\ \forall v_i \notin S_1$. Therefore $D = S_1$ is minimum dominating set and $\gamma = |S_1| = 7$.

3.2 Example

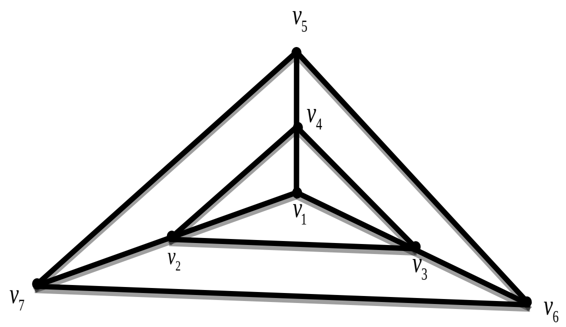

Fig 3: Graph without any support

1. Let $G = (V_G, E_G)$ be the graph as given Fig. 3 such that $n = 7$ and $t = 0$.

2. Since there is no support in G therefore we construct the set S by taking the vertex having maximum domination power.

Now $dp(v_1) = 4,\ dp(v_2) = 5,\ dp(v_3) = 5,\ dp(v_4) = 5,\ dp(v_5) = 4,\ dp(v_6) = 4,\ dp(v_7) = 4.$

Since the vertex v_2, v_3 and v_4 have maximum domination power, therefore we can take any vertex out of v_2, v_3 and v_4 as an element of S .

3. Let us take v_2 in S that is $S = \{v_2\}$ and $dp(v_2) = 5$ that is $k_1 = 5$.

4. Since $k_1 = 5 < n$ so we construct another set S_1 by extending set S.

Now $dp_{v_2}(v_1) = 0,\ dp_{v_2}(v_3) = 1,\ dp_{v_2}(v_4) = 1,\ dp_{v_2}(v_5) = 2,$ $dp_{v_2}(v_6) = 2,\ dp_{v_2}(v_7) = 2.$

Since the vertex v_5, v_6 and v_7 have same maximum relative domination power with respect to v_2 therefore we can take anyone out of v_5, v_6 and v_7 in set S_1.

5. Suppose we take v_5 in S_1 that is $S_1 = \{v_2, v_5\}$ and $dp_{v_2}(v_5) = 2$, so $k_2 = 2$.

6. Now $k_1 + k_2 = 7 = n,$ and $dp_{v_2 v_5}(v_i) = 0; \forall v_i \notin S_1$.

Therefore $D = S_1 = \{v_2, v_5\}$ is a minimum dominating set and $\gamma = |S_1| = 2$.

4. MAIN RESULTS

4.1 Theorem

If S be the set of all supports of a connected graph G, then domination number $\gamma \geq |S|$.

Proof: Let G be a connected graph of order n and S be the set of all supports in G such that $|S| = k$. Then there are at-least k pendent vertices in G. Pendent vertices are dominated by either support or by itself. Therefore either each support or corresponding pendent vertex must be in dominating set $D \geq k$. Thus $\gamma \geq k = |S|$.

4.2Theorem

If I be the set of all internal vertices of a tree T then connected domination number $\gamma_c = |I|$.

Proof: Let T be a tree of order n and let $|I| = m$ that is there are m internal vertices in T. Obviously $< I >$ is a sub tree of T and $|\overline{N}(I)| = n$. Hence I is a connected dominating set of a tree T.

I is the smallest connected dominating set of tree T. If it is not so, let I' be another connected dominating set of tree T such that $I' \subset I$. Clearly $\exists$ at least one $v_i \in I$ such that $v_i \notin I'$.

Now two cases arise:

(i) When $v_i \in I$ and v_i is a support of T such that $v_i \notin I'$. In this case $|\overline{N}(I')| < n$, so I' is not a dominating set of tree T.

(ii) When $v_i \in I$ and v_i is not a support of T such that $v_i \notin I'$. Clearly in this case $< I' >$ is not connected.

Thus there exist no proper subset of I which is connected dominating set for T. Hence I is the smallest connected dominating set for T and $\gamma_c = |I|$.

4.3 Example

Consider the Fig. 4 of labelled tree with 13 vertices as given below:

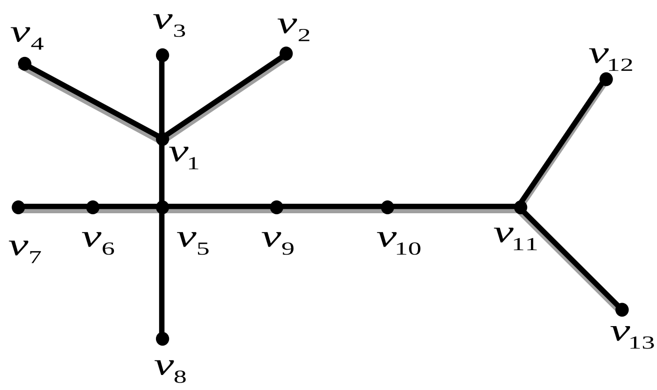

Fig 4: Labelled tree with six internal vertices and seven pendent vertices

Here $I = \{v_1, v_6, v_5, v_9, v_{10}, v_{11}\}$ and $\gamma_c = 6 = |I|$.

In the next lemma a relation between connected domination number and domination number for a tree has been established.

4.4 Lemma

If domination number γ of a tree T is equal to number of support, then

$$\gamma_c = \gamma + k,$$

where k is the number of internal vertices which are not support.

Proof: Let S be the set of all support of a tree T such that domination number $\gamma = |S|$. Let k be the number of internal vertices of T which are not support then

γ_c = number of internal vertices

γ_c = number of support + number of internal vertices which are not support.

$$\gamma_c = |S| + k$$

$$\gamma_c = \gamma + k. \qquad (\because \gamma = |S|)$$

4.5 Example

In the following tree $\gamma = 3,\ \gamma_c = 4,$ and v_2 is only internal vertex which is not support that is $k = 1$ which verify the above result

$$\gamma_c = \gamma + k.$$

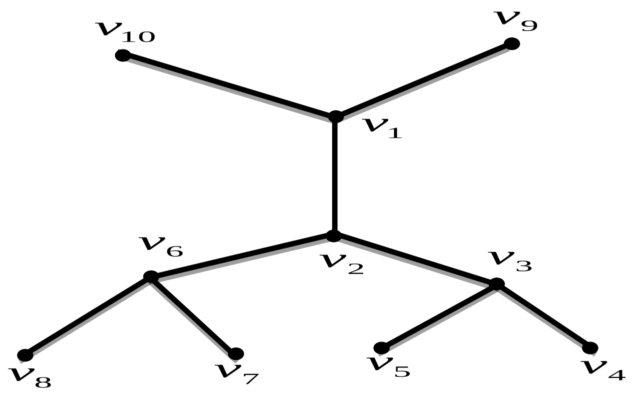

Fig 5: Tree with one internal vertex without support

4.6 Corollary

If every internal vertex of a tree is support, then domination number and connected domination number of the tree are same.

Proof: Consider a tree T with domination number γ and connected domination number γ_c. Let k be the internal vertices but not support then $\gamma_c = \gamma + k$.

Since every internal vertex of T is support that is there is no internal vertex which is not support therefore $k = 0 \Rightarrow \gamma_c = \gamma$.

Note: This corollary is also proved by Arumugam et al [2] as a theorem.

4.7 Example

Consider the tree in Fig.-6 as given below:

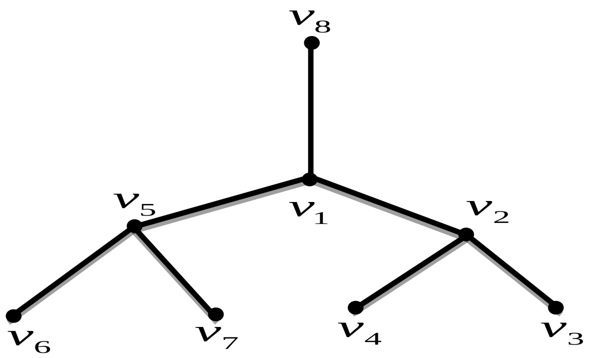

Fig 6: Tree with all internal vertices are support

Here v_1, v_2 and v_5 are three internal vertices which all are support and

$$\gamma = 3 = \gamma_c.$$

4.8 Theorem

If T be a tree with m pendent vertices then independence number β of T is $m + l$, where l is maximum number of internal vertices such that $\forall v_i, v_j \in I - S$ and v_i and v_j are not adjacent.

Proof: Let E be the largest set of independent vertices of a tree T such that $p_1, p_2, p_3, ..., p_m$ are m pendent vertices, $s_1, s_2, s_3, ..., s_t$ are t supports and $i_1, i_2, i_3 ..., i_k$ are k internal vertices except support of T, then $m \geq t$. Since each pendent vertex is adjacent only support and $m \geq t$. Therefore $\forall p_i \in E; \quad 1 \leq i \leq m$ and $\forall s_i \notin E; \quad 1 \leq i \leq t$.

Now out of k internal vertices which are not support let maximum l internal vertices are independent to each other. Suppose that these vertices are $i_1, i_2, i_3, ..., i_l$ then $\forall i_j \in E; \ \ 1 \leq j \leq l.$ So $E = \{p_1, p_2, p_3, ..., p_m; i_1, i_2, i_3, ..., i_l\}$ is the largest set of independent vertices for T and hence

$$\beta = |E| = m + l.$$

4.9 Example

In Fig. 4; $v_2, v_3, v_4, v_7, v_8, v_{12}, v_{13}$ are pendent vertices that is $m = 7$ and v_9, v_{10} are two internal vertices except support but these two vertices are adjacent so $l = 1$ and

$$\beta = 8 = m + l.$$

4.10 Corollary

If every internal vertex is support in a tree T, then independence number β of T is equal to number of pendent vertices in T.

Proof: Let T be a tree with m pendent vertices then independence number of T is given by

$$\beta = m + l,$$

where l is maximum number of internal vertices such that $\forall v_i, v_j \in I - S$ and v_i and v_j are not adjacent. Since every internal vertex is support therefore $l = 0$ that is $\beta = m =$ number of pendent vertices.

4.11 Example

Consider the tree in Fig.-6.Here v_1, v_2 and v_5 are three internal vertices which all are support and v_3, v_4, v_6, v_7 and v_8 are 5 pendent vertices that is $m = 5$ and $\beta = 5 = m$ which verify the result.

In the next theorem, a relation among domination number, connected domination number and independence number for a tree has been established.

4.12 Theorem

If in a tree T, every internal vertex except support are independent to each other then

$$\beta + \gamma = \gamma_c + m.$$

Proof: Let T be a tree such that every internal vertex except support are independent to each other than by result of corollary 4.6,

$$\gamma_c = \gamma + k, \qquad (1)$$

where k is the number of internal vertices which are not support. By result of theorem 4.8,

$$\beta = m + l, \qquad (2)$$

where l is maximum number of internal vertices such that $\forall v_i, v_j \in I - S$ and v_i and v_j are not adjacent. Since every internal vertex except support are independent to each other therefore $k = l$, using in equation (2) gives

$$\beta = m + k \Rightarrow k = \beta - m \quad (3)$$

using equation (3) in equation (1) gives

$$\gamma_c = \gamma + \beta - m \Rightarrow \beta + \gamma = \gamma_c + m .$$

5. ACKNOWLEDGEMENT

This research work is supported by Gurukula Kangri University, Haridwar (UK), India. The authors are grateful to Dr. S. R. Verma for his valuable suggestions to improve the manuscript of this paper.

6. REFERENCES

[1] Harary, F., 1997 Graph Theory, Narosa Publishing House.

[2] Arumugam, S., Joseph, J. P. 1999 On graphs with equal domination and connected domination numbers, Discrete Mathematics vol.206, 45-49.

[3] Deo, N. 2005 Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall of India Private Limited.

[4] Saoud, M., Jebran, J. 2009 Finding A Minimum Dominating Set by Transforming Domination of Vertices, Acta Universitatis Apulensis 19.

[5] Hedetniemi, S.T., Laskar, R.C. 1990 Bibliography on Domination in Graph and Some Basic Definitions on domination Parameters, Discrete Mathematics vol.86, 257-277.