# A Wheel $1$-Safe Petri Net Generating all the $\{0,1\}^n$ Sequences

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## ABSTRACT

Petri nets are a graphic and mathematic modeling tool which is applicable to several systems and to all those systems presenting particular characteristics such as concurrency, distribution, parallelism, non-determinism and/or stochastically. In this paper, a *wheel Petri net* whose reachability tree contains all the binary $n$-tuples or sequences as marking vectors has been defined. The result is proved by the using of the Principle of Mathematical Induction (PMI) on $|P| = n$.

### Keywords:

$1$-safe Petri net, reachability tree, binary $n$-vector, marking vector, wheel graph

## 1. INTRODUCTION

A Petri net is a graphical tool invented by Carl Adam Petri [1]. Its origin can be traced back to August 1939 when, at the age of 13, Petri created the graphics to describe chemical processes that produced a final compound from various elements through some intermediate compounds. The 'net-like' representation of these logical tools came into the existence in his doctoral thesis "Communication with Automata" at the Technical University of Darmstadt, Germany, in 1962 [1]. Petri nets are very reliable tool to model and study the structure of the discrete event-driven systems with large population or heavy traffic appear frequently in many fields such as manufacturing processes, logistics, telecommunication systems, traffic systems etc [2]. Of all existing models, Petri nets and their extensions are of undeniable fundamental interest because they define easy graphical support for the representation and the understanding of basic mechanism and behaviors. In the conclusion of [3], Kansal *et al.* has been shown the existence of a 1-safe Petri net which looks like a wheel, with three places and four transitions generating all the $2^3$ binary 3-vectors as marking vectors. The aim of this paper is to define, in general, a *wheel Petri net* with $|P| = n$ and $|T| = n+1$, which is obtained by subdividing the edges of a wheel graph $W_{n+1}$ for $(n+1)$ vertices and $2n$ edges. At the initial marking vector $\mu^0(p) = 1, \forall p \in P$, the reachability tree of such a Petri net contains all the all the binary $n$-vectors as a marking vectors. The result is provd by using of the Principle of Mathematical Induction on $|P| = n$. These $\{0,1\}^n$ sequences can be used to form a complete Boolean hypercube which is the most popular interconnection network with many attractive and well known properties such as regularity, symmetry, strong connectivity, expendability, recursive construction and closely related to planning formalisms, etc.

## 2. PRELIMINARIES

For standard terminology and notation on Petri nets theory and Graph theory, the reader is referred to Peterson[4] and Harary[5], respectively. Throughout this paper, the following definition given by Jensen [6] is being used.

A *Petri net* is a 5-tuple $N = (P, T, I^-, I^+, \mu^0)$, where

(1) $P$ is a nonempty set of 'places',
(2) $T$ is a nonempty set of 'transitions',
(3) $P \cap T = \emptyset$,
(4) $I^-, I^+ : P \times T \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, are called the *negative* and the *positive* 'incidence functions' (or, 'flow functions') respectively,
(5) $\forall\ p \in P, \exists t \in T : I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$ and $\forall\ t \in T, \exists p \in P : I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$,
(6) $\mu^0 : P \to \mathbb{N}$ is the *initial marking*.

In fact, $I^-(p,t)$ and $I^+(p,t)$ represent the number of arcs from $p$ to $t$ and $t$ to $p$ respectively. $I^-$, $I^+$ and $\mu^0$ can be viewed as matrices of size $|P| \times |T|$, $|P| \times |T|$ and $|P| \times 1$, respectively.

As in many standard books (e.g., see [7]), Petri net is a particular kind of directed graph, together with an initial marking $\mu^0$. The underlying graph of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called places and transitions, where arcs are either from a place to a transition or from a transition to a place. Hence, Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting places and transitions to represent the flow relation. The initial marking is represented by placing a token in the circle representing a place $p_i$ as a black dot whenever $\mu^0(p_i) = 1,\ 1 \leq i \leq n = |P|$. In general, a *marking* $\mu$ is a mapping $\mu : P \longrightarrow \mathbb{N}$. A marking $\mu$ can hence be represented as a vector $\mu \in \mathbb{N}^n,\ n = |P|$, such that the

$i^{th}$ component of $\mu$ is the value $\mu(p_i)$.

In a Petri net $N$, a transition $t \in T$ is said to be *enabled* at $\mu$ if and only if $I^-(p,t) \leq \mu(p), \forall p \in P$. An enabled transition may or may not 'fire' (depending on whether the event actually takes place or not). After firing at $\mu$, the new marking $\mu'$ is given by the rule

$$\mu'(p) = \mu(p) - I^-(p,t) + I^+(p,t), forall\ p \in P.$$

and write $\mu \xrightarrow{t} \mu'$, whence $\mu'$ is said to be *directly reachable* from $\mu$. Hence, it is clear, what is meant by a sequence like

$$\mu^0 \xrightarrow{t_1} \mu^1 \xrightarrow{t_2} \mu^2 \xrightarrow{t_3} \mu^3 \cdots \xrightarrow{t_k} \mu^k,$$

which simply represents the fact that the transitions

$$t_1, t_2, t_3, \ldots, t_k$$

have been successively fired to transform the marking $\mu^0$ into the marking $\mu^k$. The whole of this sequence of transformations is also written in short as $\mu^0 \xrightarrow{\sigma} \mu^k$, where $\sigma = t_1, t_2, t_3, \ldots, t_k$.

A marking $\mu$ is said to be *reachable from* $\mu^0$, if there exists a sequence of transitions which can be successively fired to obtain $\mu$ from $\mu^0$. The set of all markings of a Petri net $N$ reachable from a given marking $\mu$ is denoted by $\mathcal{M}(N, \mu)$ and, together with the arcs of the form $\mu^i \xrightarrow{t_r} \mu^j$, represents what in standard terminology called the *reachability graph* $R(N, \mu)$ of the Petri net $N$. If the reachability graph has no cycle then it is called *reachability tree*.

A place in a Petri net is *safe* if the number of tokens in that place never exceeds one. A Petri net is *safe* if all its places are safe.

The *preset* of a transition $t$ is the set of all input places to $t$, i.e., $^\bullet t$=$\{p \in P : I^-(p,t) > 0\}$. The *postset* of $t$ is the set of all output places from $t$, i.e., $t^\bullet$=$\{p \in P : I^+(p,t) > 0\}$. Similarly, $p's$ preset and postset are $^\bullet p$=$\{t \in T : I^+(p,t) > 0\}$ and $p^\bullet$=$\{t \in T : I^-(p,t) > 0\}$, respectively.

## 3. SOME DEFINITIONS

In this section, a wheel Petri net has been defined with the help of wheel graph [5].

DEFINITION 1. *[8] A pair of a place $p$ and a transition $t$ is called a self-loop in a Petri net if $p$ is both an input and output place of $t$ (see. Figure 1).*

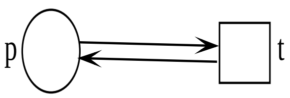


Figure 1: Self-loop between a place $p$ and a transition $t$.

DEFINITION 2. *A **wheel graph** with $n + 1$ vertices and $2n$ edges contains a cycle of length $n$ and for every graph vertex in the cycle is connected to one other graph vertex (known as the hub), it is denoted by $W_{n+1}$. The edges of a wheel which include the hub are called spokes. In other words, wheel graph can be defined as the graph $K_1$ + $C_n$, where $K_1$ is the singleton graph and $C_n$ is the cycle graph with $n$ vertices and $n$ edges. (See Figure 2).*

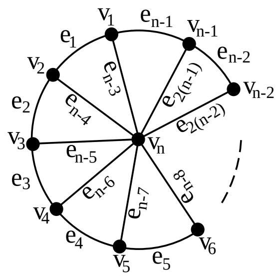


Figure 2: A wheel graph which $n + 1$-vertices and $2n$ edges.

DEFINITION 3. *A **wheel Petri net** is obtained by subdividing every edge of the cycle $C_n$ in wheel graph $W_{n+1}$, so that every subdividing vertex in the cycle graph $C_n$ together with the original singleton graph $K_1$ are the transition nodes $t_1$, $t_2$, $t_3$, $\cdots$, $t_n$, $t_{n+1}$, respectively and the original vertices of $C_n$, are the place nodes $p_1$, $p_2$, $p_3$, $\cdots$, $p_n$, respectively. Further, every arc incident to the singleton graph $K_1$ is directed towards it from the $n$ places in $C_n$, and every arc of the type $(p_i, t_i)$, $i = 1, 2, \cdots, n$ in $C_n$ is joined by a self-loop and the arc $(p_{i+1}, t_i)$ directed towards the transition $t_i$, and wheel Petri net is denoted by $N_{W_n}$. The general configuration of a wheel Petri net is shown in Figure 3.*

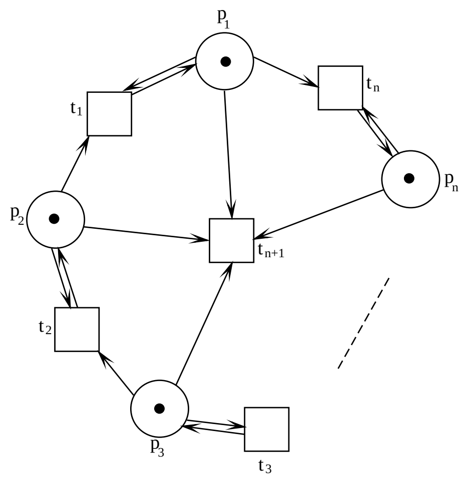


Figure 3: A wheel 1-safe Petri net $N_{W_n}$

REMARK 1. *In this paper, wheel graph $W_{n+1}$ has $n + 1$ vertices and $2n$ edges but the wheel Petri net $N_{W_n}$ has $|P| = n$ places and $n + 1$ transitions. However, in this paper, for $n = 2, 3$ vertices, I assumed that $W_2$=$K_1$ + $C_1$ trivially has a loop at the cycle graph $C_1$. In this case, after subdivision of edges in $W_2$=$K_1$ + $C_1$, the wheel Petri net $N_{W_1}$ will have one place and two transition such that one place and one transition will form a self-loop and other transition will be a sink transition. Similarly, the wheel graph $W_3$=$C_2$+$K_1$, edge in the cycle consider as symmetric pair of edges and the end vertices in $C_2$ is connected to $K_1$. In this way, wheel Petri net $N_{W_2}$ will have two places and three transitions such that $|^\bullet t_i| = 2$ and $|t_i^\bullet| = 1$, $i = 1, 2$, except the sink transition $t_3$.*

## 4. WHEEL PETRI NET IS BOOLEAN

In this section, the proof of the theorem for a wheel Petri net that generates all the $2^n$ binary $\{0,1\}^n$ sequences as marking vectors has been proved by using the Principle of mathematical Induction (PMI) on $|P| = n$.

THEOREM 1. *The reachability tree of* $N_{W_n}$ *with* $\mu_0 = (1,1,1,\ldots,1)$ *as the initial marking contains every binary n-vector* $(a_1, a_2, a_3, \ldots, a_n)$, $a_i \in \{0,\ 1\}$.

PROOF. First, let $n = 1$. After firing of the transitions $t_1$ and $t_2$, the reachability tree $R(N_{W_1}, \mu^0)$ of wheel Petri net $N_{W_1}$ contains all the $2^1$ binary 1-vectors, namely $(1)$, $(0)$ (see Figure 4 and Figure 5). Further, in the Figure 5, the new marking vector $(1)$, transitions $t_1$ and $t_2$ are further enabled and fire. To avoid the repetition of binary 1-vectors, we stop the firing of transitions.

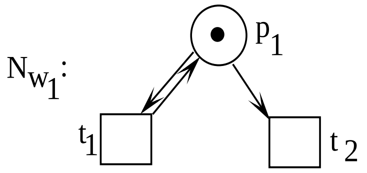


Figure 4: A wheel 1-safe Petri net $N_{W_1}$

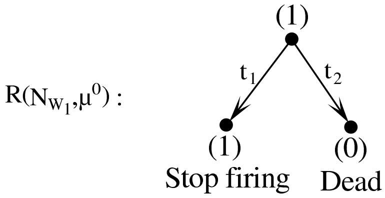


Figure 5: Reachability tree of $R(N_{W_1}, \mu^0)$

Next, let $n = 2$. Then $N_{W_2}$ contains the following structure, shown in Figure 6. Since $\mu^0(p) = 1\ \forall\ p \in P$, transitions $t_1$, $t_2$ and $t_3$ are enabled. After firing, the marking vectors $(1,0)$, $(0,1)$ and $(0,0)$, respectively are obtained. Further, at these new marking vectors, all the transitions become dead (see Figure 7). In this way, the wheel Petri net $N_{W_2}$ has all the $2^2$ binary 2-vectors as marking vectors.

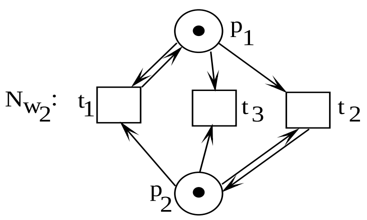


Figure 6: A wheel 1-safe Petri net $N_{W_2}$

One can obtain $N_{W_2}$ from $N_{W_1}$ and $R(N_{W_2}, \mu^0)$ from $R(N_{W_1}, \mu^0)$ procedurally as follows:

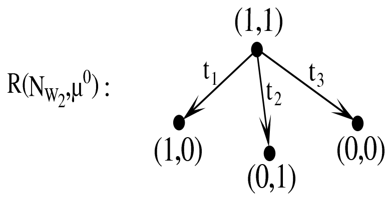


Figure 7: Reachability tree of $R(N_{W_2}, \mu^0)$

**To obtain $N_{W_2}$ from $N_{W_1}$ using the following steps**

To obtain procedurally $N_{W_2}$ from $N_{W_1}$ is the trivial case here.

step-1. Take two copies of $N_{W_1}$. In the second copy of $N_{W_1}$ remove the dotted encircle transition $t_2$ as shown in Figure 8 (do not remove arc incident (i.e., incoming or outgoing) on $t_2$).

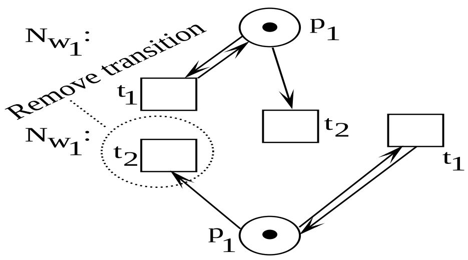


Figure 8: Before removing the transition $t_1$ and $t_2$

step-2. Join the incident arcs (incoming or outgoing) on $t_2$ in the second copy of $N_{W_1}$ to the transition $t_1$ in the first copy of $N_{W_1}$ and join the place $p_1$ in the first copy of $N_{W_1}$ to the transition $t_1$ in the second copy of $N_{W_1}$ such that each transitions must have two incoming arcs and one outgoing arc incident on them except the sink transition. Next, in the second copy of $N_{W_1}$, relabel the place $p_1$ as $p_2$ and the transition $t_1$ as $t_2$.

step-3. Next, join the relabel place $p_2$ in the second copy of $N_{W_1}$ to the sink transition $t_2$ in the first copy of $N_{W_1}$ and relabel this sink transition $t_2$ as $t_3$ (See Figure 8).

In this way, we obtain a resulting wheel 1-safe Petri net $N^*_{W_2} =: N_{W_2}$ (See Figure 9).

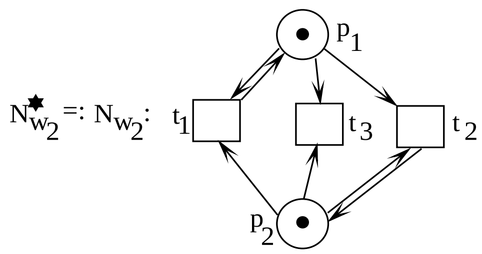


Figure 9: After joining the arcs, the resulting wheel Petri net $N^*_{W_2} =: N_{W_2}$

**To obtain $R(N_{W_2}, \mu^0)$ from $R(N_{W_1}, \mu^0)$ using the following steps**

To obtain procedurally $R(N_{W_2}, \mu^0)$ from $R(N_{W_1}, \mu^0)$ is the trivial case here.

Using the following steps we construct the reachability tree $R(N_{W_2}, \mu^0)$ of $N_{W_2}$ from $R(N_{W_2}, \mu^0)$ of $R(N_{W_1}, \mu^0)$.

step-(i). Take two copies of $R(N_{W_1}, \mu^0)$. In the first copy, augment each vector of $R(N_{W_1}, \mu^0)$, by putting a '1′ entry at the second position of every marking vector and denote the resulting labeled tree as $R_1(N_{W_1}, \mu^0)$ (See Figure 10). Similarly, in the second copy, augment each vector by putting '0′ at the second position of every marking and let $R_0(N_{W_1}, \mu^0)$ be the resulting labeled tree (See Figure 11).

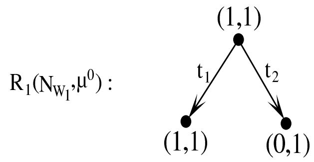


Figure 10: Augmented reachability tree $R_1(N_{W_1}, \mu^0)$.

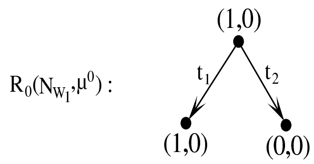


Figure 11: Augmented reachability tree $R_0(N_{W_1}, \mu^0)$.

step-(ii). Clearly, the set of binary 2-vectors in $R_1(N_{W_1}, \mu^0)$ is disjoint with the set of those appearing in $R_0(N_{W_1}, \mu^0)$ and together they contain all the binary 2-vectors.

step-(iii). In $R_1(N_{W_1}, \mu^0)$, transition $t_1$ is enabled and the marking obtained after firing of $t_1$ is actually $(1, 0)$ whereas the augmented vector attached to this node is $(1, 1)$. So, we concatenate $R_0(N_{W_1}, \mu^0)$ by fusing the root node labeled as $(1, 0)$ with the augmented child node labeled as $(1, 1)$ in $R_1(N_{W_1}, \mu^0)$ and replacing $(1, 1)$ by the label $(1, 0)$ which is the initial marking of $R_0(N_{W_1}, \mu^0)$. Further, at this marking vector $(1, 0)$ all the transitions become dead therefore we delete all the child nodes of it together with marking vectors and arcs.

step-(iv). Now then augment an extra pendent node labeled $y_0$ by the 2-vector $(0, 0)$ joined to the new root node $x_0$ labeled by the 2-vector $(1, 1)$ by the new arc $(x_0, y_0)$ labeled as $t_3$. The resulting labeled tree $T^*$ has all the binary 2-vectors as its node labels. It remains to show that $T^*$ is the reachability tree $R(N_{W_2}, \mu^0)$ of $N_{W_2}$ with 2-vector $(1, 1)$ as its initial marking $\mu^0$. For this, consider an arbitrary 2-vector $\mu = (a_1, 1)$, where $a_1 \in \{0, 1\}$. When $a_1 = 0$ then no one transitions are enabled and fire i.e., all are become dead. This can be seen in $T^*$ easily (See Figure 12). If $a_1 = 1$ then all the transitions $t_1$, $t_2$ and $t_3$ are enabled and fire, this yields

$$\mu'(p_i) = \mu(p_i) - I^-(p_i,\ t_j) + I^+(p_i,\ t_j), j = 1, 2, 3.$$

$$= 1 - 1 + 0 = 0$$

Then, the new markings vectors are $(1, 0)$,$(0, 1)$ and $(0, 0)$ respectively. The marking $(1, 0)$,$(0, 1)$ and $(0, 0)$ are found in $R(N_{W_2}, \mu^0)$ of $N_{W_2}$. Hence, suppose some $a_i = 0$. In this case, $t_i$ is not enabled. Eventually, this process will lead to a dead marking. Further, the marking vectors $(1, 0)$,$(0, 1)$ and $(0, 0)$ are already obtained as a result of firing $t_1, t_2, t_3$ in the first stage of firing. Thus, $T^*$ is indeed the reachability tree $R(N_{W_2}, \mu^0)$ of $N_{W_2}$ (See Figure 12).

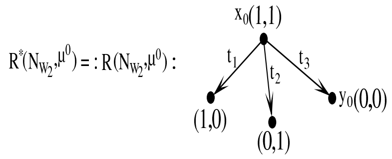


Figure 12: $T^*$=: $R(N_{W_2}, \mu^0)$.

Further, let $n = 3$. Then $N_{W_3}$ contains the following structure as shown in Figure 13. Since $\mu^0(p) = 1 \ \forall \ p \in P$, all the transitions $t_1$, $t_2$, $t_3$ and $t_4$ are enabled. After firing of these transitions in the first stage, the reachability tree $R(N_{W_3}, \mu^0)$ of $N_{W_3}$ has ${}^3C_1 + 1$ marking vectors of Hamming distance 1 from $\mu^0$ which are $(1, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$ and $(0, 0, 0)$ respectively. Further, at the marking vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 1, 1)$ the transitions $t_3$, $t_1$ and $t_2$ are enabled. After firing we get the marking vectors of Hamming distance 2 from $\mu^0$ which are $(0, 0, 1)$, $(1, 0, 0)$ and $(0, 1, 0)$ respectively and at the marking vector $(0, 0, 0)$ all the transitions become dead. Next, the marking vectors obtained in the second stage, no one transitions is enabled. In this way, $R(N_{W_3}, \mu^0)$ of $N_{W_3}$ contains all the $2^3 = 8$ binary 3-vectors as marking vectors (See Figure 14). Therefore, $N_{W_3}$ has all the $2^3$ binary 3-vectors as marking vectors.

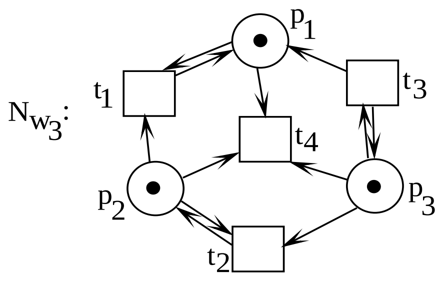


Figure 13: A wheel 1-safe Petri net $N_{W_3}$

Next we can obtain $N_{W_3}$ from $N_{W_2}$ and $R(N_{W_3}, \mu^0)$ from $R(N_{W_2}, \mu^0)$ procedurally as follows:

**To obtain $N_{W_3}$ from $N_{W_2}$ using the following steps**

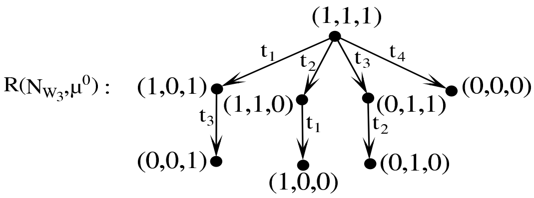


Figure 14: Reachability tree of $R(N_{W_3}, \mu^0)$

step-1. Take one copy of $N_{W_2}$ and one copy of $N_{W_1}$. In $N_{W_2}$ remove the dotted encircle transition $t_2$ (do not remove arcs incident(incoming or outgoing) as shown in the Figure 15.

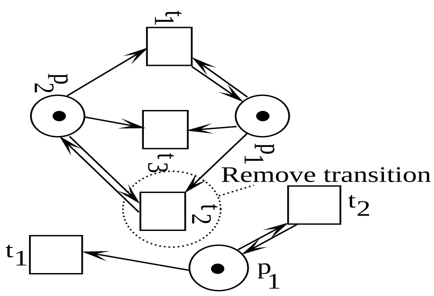


Figure 15: Before removing the transition $t_2$.

step-2. Join the incident arcs (incoming or outgoing) on $t_2$ in the first step to the transitions of $t_1$ and $t_2$ in the $N_{W_1}$ such that each $|^\bullet t_1| = 2$ and $|t_2^\bullet| = 1$. Further, in $N_{W_1}$ labeled the transitions $t_1$, $t_2$ as $t_2$, $t_3$ respectively and place $p_1$ labeled as $p_3$.

step-3. Next, join the relabeled place $p_3$ in $N_{W_1}$ to the sink transition $t_3$ of $N_{W_2}$ and labeled the transition $t_3$ as $t_4$.

In this way, from all the steps, we obtain a resulting wheel 1-safe Petri net $N^*_{W_3} =: N_{W_3}$ (See Figure 16).

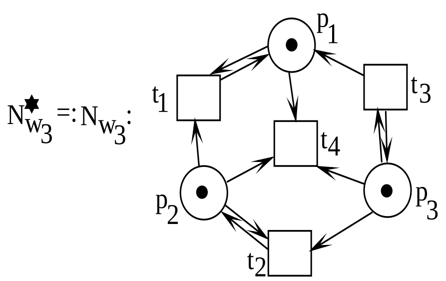


Figure 16: After joining the arcs, the resulting wheel Petri net $N^*_{W_3} =: N_{W_3}$

**To obtain $R(N_{W_3}, \mu^0)$ from $R(N_{W_2}, \mu^0)$ using the following steps**

Using the following steps we construct the reachability tree $R(N_{W_3}, \mu^0)$ of $N_{W_3}$ from $R(N_{W_2}, \mu^0)$ of $N_{W_2}$.

step-(i). Take two copies of $R(N_{W_2}, \mu^0)$. In the first copy, augment each vector of $R(N_{W_2}, \mu^0)$, by putting a '1′ entry at the third position of every marking vector and denote the resulting labeled tree as $R_1(N_{W_2}, \mu^0)$ (See Figure 17). Similarly, in the second copy, augment each vector by putting '0′ at the third position of every marking and let $R_0(N_{W_2}, \mu^0)$ be the resulting labeled tree (See Figure 18).

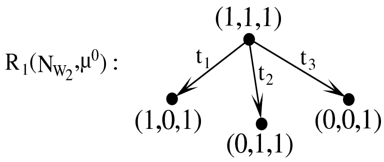


Figure 17: Augumented reachability tree $R_1(N_{W_2}, \mu^0)$.

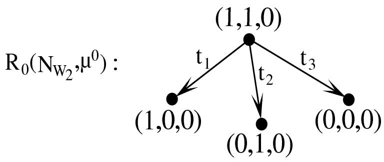


Figure 18: Augumented reachability tree $R_0(N_{W_2}, \mu^0)$.

step-(ii). Clearly, the set of binary 3-vectors in $R_1(N_{W_2}, \mu^0)$ is disjoint with the set of those appearing in $R_0(N_{W_2}, \mu^0)$ and together they contain all the binary 3-vectors.

step-(iii). In $R_1(N_{W_2}, \mu^0)$, transition $t_2$ is enabled and the marking obtained after firing of $t_2$ is actually $(1, 1, 0)$ whereas the augmented vector attached to this node $(0, 1, 1)$ is the resulting marking of the transition $t_3$. So, we replace the augmented marking node $(0, 0, 1)$ by the actual marking node $(0, 1, 1)$ and concatenate $R_0(N_{W_2}, \mu^0)$ by fusing the root node labeled as $(1, 1, 0)$ with the augmented child node labeled as $(0, 1, 1)$ in $R_1(N_{W_2}, \mu^0)$ and replacing $(0, 1, 1)$ by the label $(1, 1, 0)$ which is the initial marking of $R_0(N_{W_2}, \mu^0)$. Further, at this marking vector $(1, 1, 0)$ only the transitions $t_1$ is enabled and fire which will give the resulting marking vector $(1, 0, 0)$ and remaining transitions $t_2$ and $t_3$ are not enabled therefore we delete all the child nodes of it together with marking vectors (i.e., $(0, 1, 0)$ and $(0, 0, 0)$) and arcs. Further, in the $R_1(N_{W_2}, \mu^0)$ the marking vector $(1, 0, 1)$obtain after the firing of $t_1$ at $(1, 1, 1)$ enables the transition $t_3$ which give after firing $(0, 0, 1)$. So join this marking vector $(0, 0, 1)$ by an directed arc from the root node $(1, 0, 1)$ of $R_1(N_{W_2}, \mu^0)$ to the node of the resulting marking $(0, 0, 1)$. Similarly, complete the firing at replacing child node $(0, 1, 1)$ in $R_1(N_{W_2}, \mu^0)$.

step-(iv). We then augment an extra pendent node labeled $y_0$ by the 3-vector $(0,0,0)$ joined to the new root node $x_0$ labeled by the 3-vector $(1,1,1)$ by the new arc $(x_0, y_0)$ labeled as $t_4$. The resulting labeled tree $T^*$ has all the binary 2-vectors as its node labels. It remains to show that $T^*$ is the reachability tree $R(N_{W_3}, \mu^0)$ of $N_{W_3}$ with $\mu^0$=$(1,1,1)$ as its initial marking . For this, consider an arbitrary 3-vector $\mu = (a_1, a_2, 1)$, where $a_1, a_2 \in \{0,1\}$. Then the following cases are arise.

(1) If both $a_1, a_2$ are one then all the transition are enabled. After firing of $t_1$, $t_2$, $t_3$ and $t_4$ in the first stage we get $(1,0,1)$ $(1,1,0)$, $(0,1,1)$ and $(0,0,0)$ respectively which are already present in $T^*$. After that only single transition is enabled at each marking except the zero vector.

(2) If any one of $a_1, a_2$ are zero then after firing of $t_3$, $t_1$, $t_2$ at the marking $(1,0,1)$ $(1,1,0)$, $(0,1,1)$ we get $(0,0,1)$ $(1,0,0)$, $(0,1,0)$ respectively. These are also present in the $T^*$.

(3) If both $a_1, a_2$ are zero then all the transitions become dead which also case in the $T^*$.

This can be seen in $T^*$ easily (See Figure 19). Thus, $T^*$ is indeed the reachability tree $R(N_{W_3}, \mu^0)$ of $N_{W_3}$. Hence $N_{W_3}$ contains all the $2^3 = 8$ binary 3-vectors as marking vectors.

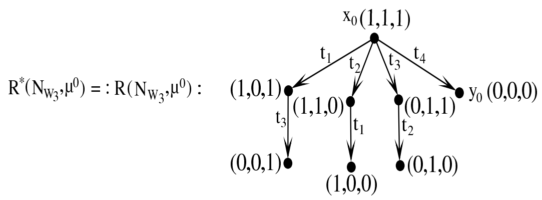


Figure 19: $T^*$=: $R(N_{W_3}, \mu^0)$.

Now, we assume that the result is true for all the wheel 1-safe Petri nets $N_{W_k}$ having $k$-places, $k \leq n$. We will prove the result for the wheel 1-safe Petri net $N_{W_{k+1}}$ having $(k+1)$ places.

**To obtain $N_{W_{k+1}}$ from $N_{W_k}$ using the following steps**

step-1. Take one copy of $N_{W_k}$ and one copy of $N_{W_1}$. In $N_{W_k}$ remove the transitions $t_k$ (do not remove arcs incident (incoming or outgoing) on $t_k$

step-2. Join the incident arcs (incoming or outgoing) on $t_k$ in the first step to the transitions $t_1$ and $t_2$ in the copy of $N_{W_1}$ such that $|{}^\bullet t_1| = 2$ and $|t_2^\bullet| = 1$. Further, in $N_{W_1}$ relabel the place $p_1$ as $p_{k+1}$ and the transitions $t_2$ and $t_1$ as $t_k$ and $t_{k+1}$, respectively.

step-3. Next, join the place $p_{k+1}$ to the sink transition $t_{k+1}$ of $N_{W_k}$ by an arc $p_{k+1}, t_{k+1}$ and labeled this sink transition $t_{k+1}$ as $t_{k+2}$.

In this way, from all the steps, we obtain a resulting wheel 1-safe Petri net $N^*_{W_{K+1}} =: N_{W_{K+1}}$.

**To obtain $R(N_{W_{k+1}}, \mu^0)$ from $R(N_{W_k}, \mu^0)$ using the following steps**

Using the following steps, we construct the reachability tree $R(N_{W_{k+1}}, \mu^0)$ of $N_{W_{k+1}}$ from $R(N_{W_k}, \mu^0)$ of $N_{W_k}$.

step-(i). Take two copies of $R(N_{W_k}, \mu^0)$. In the first copy, augment each vector of $R(N_{W_k}, \mu^0)$, by putting a '1' entry at the $(k+1)^{th}$ position of every marking vector and denote the resulting labeled tree as $R_1(N_{W_k}, \mu^0)$. Similarly, in the second copy, augment each vector by putting '0' at the $(k+1)^{th}$ position of every marking and let $R_0(N_{W_k}, \mu^0)$ be the resulting labeled tree.

step-(ii). Clearly, the set of binary $(k+1)$-vectors in $R_1(N_{W_k}, \mu^0)$ is disjoint with the set of those appearing in $R_0(N_{W_k}, \mu^0)$ and together they contain all the binary $(k+1)$-vectors.

step-(iii). In $R_1(N_{W_k}, \mu^0)$, transition $t_k$ is enabled and the marking obtained after firing of $t_k$ is actually $(1,1,1,\cdots,0)$ whereas the augmented vector attached to this node $(0,1,1,\cdots,1)$ is the resulting marking of the transition $t_{k+1}$. So, we replace the augmented marking node $(0,0,\cdots,1)$ by the actual marking node $(0,1,1,\cdots,1)$ and concatenate $R_0(N_{W_k}, \mu^0)$ by fusing the root node labeled as $(1,1,1,\cdots,0)$ with the augmented child node labeled as $(0,1,1,\cdots,1)$ in $R_1(N_{W_k}, \mu^0)$ and replacing $(0,1,1,\cdots,1)$ by the label $(1,1,1,\cdots,0)$ which is the initial marking of $R_0(N_{W_k}, \mu^0)$. Further, at this marking vector $(1,1,1,\cdots,0)$ only the transitions $t_1$, $t_2$, $\cdots$, $t_{k-1}$ are enabled and fire which will gives the marking vectors which have the zero at $t_2^{th}$, $t_3^{th}$ $\cdots$, $t_{k-2}^{th}$ position and remaining transitions are not enabled therefore we delete all the child nodes of it together with marking nodes and arcs. Further, in the $R_1(N_{W_k}, \mu^0)$ the marking vector $(1,0,1,\cdots,1)$ obtain after the firing of $t_1$ at $(1,1,1,\cdots,1)$ enables the next $(k-1)$ transitions, we fire till all transitions become dead. Similarly, complete the reachability tree for other markings.

step-(iv). We then augment an extra pendent node labeled $y_0$ by the $(k+1)$-vector $(0,0,0,\cdots,0)$ joined to the new root node $x_0$ labeled by the $(k+1)$-vector $(1,1,1,\cdots,1)$ by the new arc $(x_0, y_0)$ labeled as $t_{k+2}$. The resulting labeled tree $T^*$ has all the binary $(k+1)$-vectors as its node labels. It remains to show that $T^*$ is the reachability tree $R(N_{W_{k+1}}, \mu^0)$ of $N_{W_{k+1}}$ with $(k+1)$-vector $(1,1,1,\cdots,1)$ as its initial marking $\mu^0$.

For this, consider an arbitrary $(k+1)$-vector $\mu = (a_1, a_2, \cdots, a_k, 1)$, where $a_1, a_2, \cdots, a_k \in \{0,1\}$. Then the following cases are arise.

(1) If both $a_1, a_2, \cdots, a_k$ are one then all the transition are enabled. After firing of $t_1$,$t_2$,$t_3$,$\cdots$,$t_{k+1}$ and $t_{k+2}$ in the first stage we get $n+2$, $(k+1)$-vectors $(1,0,1,\cdots,1)$ $(1,1,0,\cdots,1)$,$(0,1,1,\cdots,1)$,$\cdots$,$(0,1,1,\cdots,1)$ and $\cdots$,$(0,0,0,\cdots,0)$, respectively which are already present in $T^*$. After that only k-2 transitions are enabled at each marking except the zero vector.

(2) If any one of $a_1, a_2, \cdots, a_k$ are zero then after firing of transitions we have the marking vectors that are already present in the $T^*$.

(3) If all $a_1, a_2, \cdots, a_k$ are zero then all the transitions become dead which also case in the $T^*$.

Thus, $T^*$ is indeed the reachability tree of $N_{W_{k+1}}$. Hence $N_{W_{k+1}}$ contains all the $2^{k+1}$ binary $(k+1)$-vectors as marking vectors.

Hence, $N_{W_{k+1}}$ generates all the $2^{k+1}$ binary n-vectors as their marking vectors. □

## 5. CONCLUSION AND SCOPE

As a conclusion, Petri nets whose marking vectors are all $(0, 1)$-vectors are very much useful in designing of generalized cyclic multi-switches [9]. While solution to such a problem can perhaps be used gainfully in many purely theoretical areas like mathematics, computer science, universal algebra and order theory. Since these are large in numbers. Therefore, a computationally good characterization of such Petri nets with least possible order (i.e., $|P| + |T|$), least possible size (i.e., the number of arcs) and the number of enabled transitions are highly desirable.

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