

Types of Generalized Open Sets with Ideal

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ABSTRACT

The aim of this paper is to introduce and study the notions of α (resp., pre, semi, β)-open sets in terms of an ideal I . These concepts generalize the usual notions of α (resp., pre, semi, β) open sets. Also, several of their topological properties are investigated.

Keywords

Pre-open set, Semi-open set, α -open set, β -open set, Topological ideal, I -pre-open set, I -semi-open set.

1. INTRODUCTION

It is well-known that a large number of papers is devoted to study classes of subsets of a topological space, containing the class of open sets, and possessing properties more or less similar to those of open sets e.g. pre-open sets [7, 10, 11], semi-open sets [9], α -open sets [13] and β -open sets [1] (called semi-preopen sets [2]).

Relevant to the present work, the idea of using topological ideals in describing topological notions, which for some years now has been an interesting subject for investigation [see some of the pioneering works in 3, 4, 5]. We recall here that an ideal I on a topological space (X, τ) is a non-empty collection of subsets of X having the heredity property (that is, if $A \in I$ and $B \subseteq A$, then $B \in I$) and also satisfying finite additivity (that is, if $A, B \in I$, then $A \cup B \in I$).

The concept of I -semi-open and I -pre-open sets in ideal topological spaces were introduced and studied by [12] and [14], respectively. These definitions turned our attention to proceed in this direction and introduce the concept of α (resp., β)-open sets with respect to an ideal I , which is said to be I - α (resp., I - β)-open sets. These notations are extensions of usual concepts α (resp., β)-open sets. Some of their basic properties will be obtained. These new definitions enable us to establish relationships between definitions of I -semi-open, I -pre-open sets, I - α -open sets and I - β -open sets.

Throughout the current work a topological space (X, τ) (or simply X) will be utilized, where no separation axioms are assumed. The usual notations $Cl(A)$ for the closure and $Int(A)$ for the interior of a subset A of a topological space (X, τ) is used.

We recall some notions which are used in our sequel.

Definition 1.1. A subset A of a topological space (X, τ) is said to be:

(1) Pre-open set [10], if $A \subseteq Int(Cl(A))$ and a pre-closed set, if $Cl(Int(A)) \subseteq A$.

(2) Semi-open set [9], if $A \subseteq Cl(Int(A))$ and a semi-closed set, if $Int(Cl(A)) \subseteq A$.

(3) α -open set [13], if $A \subseteq Int(Cl(Int(A)))$ and a α -closed set, if $Cl(Int(Cl(A))) \subseteq A$.

(4) β -open set [1], if $A \subseteq Cl(Int(Cl(A)))$ and a β -closed set, if $Int(Cl(Int(A))) \subseteq A$.

Definition 1.2. A subset A of a topological space (X, τ) is said to be:

(1) Dense set [8], if $Cl(A) = X$.

(2) Nowhere dense set [6], if $Int(Cl(A)) = \emptyset$.

2. GENERALIZED OPEN SETS WITH RESPECT TO AN IDEAL

In this section we will discuss I - α (resp., I -pre, I -semi, I - β)-open sets in the topological space.

Definition 2.1. A subset A of X is said to be:

(1) α -open with respect to an ideal I (written as I - α -open), if there exists an open set U such that $(A - Int(Cl(U))) \in I$ and $(U - A) \in I$.

(2) Pre-open with respect to an ideal I (written as I -pre-open) [14], if there exists an open set U such that $(A - U) \in I$ and $(U - Cl(A)) \in I$.

(3) Semi-open with respect to an ideal I (written as I -semi-open) [12], if there exists an open set U such that $(A - Cl(U)) \in I$ and $(U - A) \in I$.

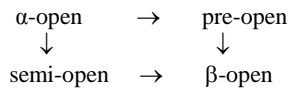
(4) β -open with respect to an ideal I (written as I - β -open), if there exists an open set U such that $(A - Cl(U)) \in I$ and $(U - Cl(A)) \in I$.

(5) α (resp., pre, semi, β)-closed set with respect to an ideal [briefly, I - α (resp., I -pre, I -semi, I - β)-closed] set iff the complement $(X - A)$ is I - α (resp., I -pre, I -semi, I - β)-open set.

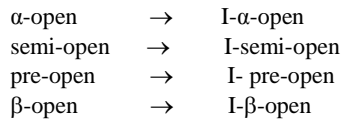
Definition 2.2. If a subset A of X is both I - α (resp., I -pre, I -semi, I - β)-open set and I - α (resp., I -pre, I -semi, I - β)-closed set, then it is called I - α (resp., I -pre, I -semi, I - β)-clopen set.

Lemma 2.3.[12] If $A \in I$, then A is I -semi-open set.

Remark 2.4.[11]



Remark 2.5.For any ideal I on X and by using Definitions 1.1 and 2.1., the following hold:

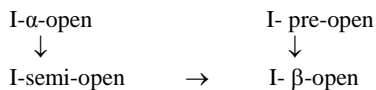


The following examples show that the reverse implications are not satisfied.

Example 2.6 Consider a topological space (X, τ) ; $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{a, c\}, X\}$. Choose $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ and observe that:

- (1) $\{b\}$ is I -pre (resp., I -semi)-open; however, it is not pre (resp., semi)-open set.
- (2) $\{a\}$ is I - α (resp., I - β)-open; however, it is not α (resp., β)-open set.

Lemma 2.7.For any ideal I on X , the following hold:



Proof In view of Definition 2.1., the proof is obvious.

Example 2.8.Consider the topological space (X, τ) ; in example 2.6 with $I = \{\emptyset, \{c\}\}$. Then, $\{b\}$ is I - α -open; and is not I -pre-open.

Example 2.9.Consider a topological space (X, τ) ; $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{a, c\}\}$ and $I = \{\emptyset, \{d\}\}$. Then:

- (1) $\{a\}$ is I -pre-open; however, it is not I - α -open set.
- (2) $\{a\}$ is I - β -open; however, it is not I -semi-open set.

Example 2.10.Consider a topological space (X, τ) ; $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $I = \{\emptyset, \{c\}\}$. Then:

- (1) $\{a\}$ is I -semi-open; however, it is not I - α -open set.
- (2) $\{a, c\}$ is I - β -open; however, it is not I -pre-open set.

According to examples 2.8 and (i) in 2.9 the concepts of I - α open set and I -pre-open set are independent although every α open set is pre-open set in [11].

Lemma 2.11.(see [4]) If A is a dense subset of (X, τ) , then it is pre-open set, moreover it is β -open set.

Corollary 2.12.If A is a dense subset of (X, τ) , then it is I -pre-open set and I - β -open set.

Proof In view of Remark 2.5., Lemma 2.7 and Lemma 2.11, the desired proof is clear.

Lemma 2.13.If A is a nowhere dense subset of (X, τ) , then it is α -closed set and semi-closed set.

Corollary 2.14 If A is a nowhere dense subset of (X, τ) and $B \subseteq A$. Then,

- (1) A is I - α -closed set and I -semi-closed set.
- (2) B is I - α -closed set and I -semi-closed set.
- (3) $Cl(A)$ is I - α -closed set and I -semi-closed set

Remark 2.15.Notice that τ -open set U is necessarily non-empty, otherwise for any I - α (resp., I -pre, I - β)-open subset A of X and $U = \emptyset$, then A is I -semi-open set.

In the next part, we discuss some of the properties of I - α (resp., I -pre, I -semi, I - β)-open sets.

Proposition 2.16.[12, 14] If A and B are both I -semi (resp., I -pre)-open set, then so is their union $A \cup B$.

Theorem 2.17.The union of two I - α (resp., I - β)-open sets of (X, τ) is also an I - α (resp., I - β)-open set.

Proof Let A and B be I - α -open sets, then $\exists U, V \in \tau$ such that $(A - \text{Int } Cl(U)) \in I$, $(U - A) \in I$, $(B - \text{Int } Cl(V)) \in I$ and $(V - B) \in I$. Choose $W = U \cup V$, hence $(U - (A \cup B)) \subseteq (U - A) \in I$, $(V - (A \cup B)) \subseteq (V - B) \in I$ and so, $(U - (A \cup B)) \cup (V - (A \cup B)) \in I$. Consequently, $(W - (A \cup B)) \in I$. Also, $(A - \text{Int } Cl(W)) \subseteq (A - \text{Int } Cl(U)) \in I$ and $(B - \text{Int } Cl(W)) \subseteq (B - \text{Int } Cl(V)) \in I$. Then, $(A - \text{Int } Cl(W)) \cup (B - \text{Int } Cl(W)) \in I$ and so $(A \cup B - \text{Int } Cl(W)) \subseteq (A - \text{Int } Cl(U)) \cup (B - \text{Int } Cl(V)) \in I$. Thus $A \cup B$ is I - α -open set. The rest of the proof is similar.

The intersection of two I - α (resp., I -pre, I -semi, I - β)-open sets need not to be an I - α (resp., I -pre, I -semi, I - β)-open set as shown by the following example.

Example 2.18.Consider a topological space (X, τ) ; $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b, d\}, \{a, d\}, \{a, b\}, \{b, d\}, \{d\}, \{b\}, \{a\}, X\}$. Choose $I = \{\emptyset, \{c\}\}$.

- (1) If $A = \{a, c\}$, $B = \{b, c\}$, then A and B are I - α -open sets but their intersection $A \cap B = \{c\}$ is not an I - α -open set.
- (2) If $A = \{a, c\}$, $B = \{b, c\}$, then A and B are I -semi-open sets but their intersection $A \cap B = \{c\}$ is not an I -semi-open set.
- (3) If $A = \{a, c\}$, $B = \{c, d\}$, then A and B are I -pre-open sets but their intersection $A \cap B = \{c\}$ is not an I -pre-open set.
- (4) If $A = \{a, c\}$, $B = \{b, c\}$, then A and B are I - β -open sets but their intersection $A \cap B = \{c\}$ is not an I - β -open set.

Theorem 2.19 (1) The intersection of I -pre (resp., I -semi, I - α , I - β)-open and open sets is an I -pre (resp., I -semi, I - α , I - β)-open set.

(2) The intersection of dense and open sets is an I-pre-open set. Moreover, it is an I- β -open set.

Proof (1) Let A be an I-pre-open set and B be an open set, then there exists an open set U such that $(A - U) \in I$ and $(U - Cl(A)) \in I$. Since $(U \cap B)$ is an open set, then $(A \cap B) - (U \cap B) = (A \cap B) - U \subseteq (A - U) \in I$. Since B is open, then $Cl(A) \cap B \subseteq Cl(A \cap B)$ and $((U \cap B) - Cl(A)) \in I$. Hence, $(U \cap B) - Cl(A \cap B) \subseteq ((U \cap B) - Cl(A) \cap B) \subseteq ((U \cap B) - Cl(A)) \in I$. Consequently, $(A \cap B)$ is an I-pre-open set. We can prove the rest of the theorem similarly.

(2) Let A be a dense set and B be an open set, then $(A \cap B) - B = \emptyset \in I$. Since $Cl(A) \cap B \subseteq Cl(A \cap B)$ and $Cl(A) = X$, then $B \subseteq Cl(A \cap B)$ and so $B - Cl(A \cap B) = \emptyset \in I$. Consequently, $(A \cap B)$ is an I-pre-open set. By using Lemma 2.7., $(A \cap B)$ is an I- β -open set.

Theorem 2.20 If A is dense and B is open in the topological space (X, τ) , then $Cl(A \cap B)$ is an I-semi-open set and I- β -open set.

Proof Let A be a dense set and B be an open set, then $Cl(A) \cap B \subseteq Cl(A \cap B)$ and $Cl(A) = X$. Hence $B \subseteq Cl(A \cap B)$ and so $B - Cl(A \cap B) = \emptyset \in I$. Also, $Cl(A \cap B) - Cl(B) = \emptyset \in I$. Consequently, $Cl(A \cap B)$ is an I-semi-open set. By using Lemma 2.7., $Cl(A \cap B)$ is an I- β -open set.

The proofs of the next theorems are obvious and so are omitted.

Theorem 2.21 Let $I = \{\emptyset\}$ on a topological space X. Then:

- (1) The concepts of α -openness and I- α -openness are the same.
- (2) The concepts of pre-openness and I-pre-openness are the same.
- (3) The concepts of semi-openness and I-semi-openness are the same.
- (4) The concepts of β -openness and I- β -openness are the same.

Theorem 2.22 Let I and J be two ideals on a topological space X. Then we have,

- (1) If $I \subseteq J$, then every I- α (resp., I-pre, I-semi, I- β)-open set is J- α (resp., J-pre, J-semi, J- β)-open set.
- (2) If a subset A is an $(I \cap J)$ - α (resp., $(I \cap J)$ -pre, $(I \cap J)$ -semi, $(I \cap J)$ - β)-open set, then it is simultaneously I- α (resp., I-pre, I-semi, I- β)-open set and J- α (resp., J-pre, J-semi, J- β)-open set.

Theorem 2.23 If B is an I-pre (resp., I- β)-open set and $A \subseteq B \subseteq Cl(A)$ in (X, τ) , then A is an I-pre (resp., I- β)-open set.

Corollary 2.24 Let A be a subset of a topological space X such that $Cl(A)$ is an I-pre (resp., I- β)-open set. Then A is an I-pre (resp., I- β)-open set for any ideal I on X.

Corollary 2.25 Let A and B be subsets of a topological space X such that B is open, $A \subseteq B$, and A is dense in B (that is, $B \subseteq Cl(A)$). Then A is an I-pre-open set for any ideal I on X.

Definition 2.26.(see [7]) A topological space (X, τ) is said to be:

- (1) Hyper connected, if every nonempty open subset of X is dense.
- (2) Locally indiscrete, if every open subset of X is closed.
- (3) Extremely disconnected, if the closure of every open subset of X is open.

Theorem 2.27. Let I be an ideal on a hyper connected topological space (X, τ) , $A \subseteq X$ and the collection of open subsets of X satisfies the finite intersection property (see [8]). Then the following statements hold:

- (1) If A is I-pre-closed set, then A is dense.
- (2) If A is I-pre-closed set, then it is I-pre-open set.
- (3) If A is I-pre-closed set and $A \subseteq B$, then B is I-pre-open set.
- (4) If A is I-pre-closed set. Then, $A \cup B$ is I-pre-open, for any subset B of X.
- (5) If A is I-pre-open and $B \subseteq A$. Then, $A \cap B$ is I-pre-closed, for any subset B of X.

Proof (1) Let A be an I-pre-closed subset of X, then there exists closed set F such that $(Int(A) - F) \in I$ and $(F - A) \in I$. Since X is hyper connected, the collection of open subsets of X satisfies the finite intersection property and $(Int(A) - F)$ is non-empty open set, then $(Int(A) - F)$ is dense and $X = Cl(Int(A) - F) \subseteq Cl(A)$ and so A is dense.

(2) In view of Lemma 2.7 and by using (1) of this theorem. Since A is an I-pre-closed set, then it is I-pre-open.

(3) Let A be I-pre-closed and $A \subseteq B$ and by using (1) of this theorem, then A is dense and so B is dense. In view of Lemma 2.7 B is I-pre-open set. The rest of the proofs are obvious.

Theorem 2.28. Let I be an ideal on a locally indiscrete topological space (X, τ) , then the following statements are hold:

- (1) Every I-semi-open is an I-pre-open.
- (2) Every I-semi-open is an I- α -open.
- (3) Every I- β -open is an I-pre-open.
- (4) Every I- α -open is an I-pre-open.
- (5) If $A \subseteq X$, then A is an I-pre-closed.

Proof We shall prove (1), (5) only and the rest of the proof is obvious.

(1) Let A be an I-semi-open, then there exists an open set U such that $(A - Cl(U)) \in I$ and $(U - A) \in I$. Since X is locally indiscrete, then A is an I-pre-open set.

(4) Let $A \subseteq X$. Choose $F = Int(A)$. Since X is locally indiscrete, then F is closed, $(Int(A) - F) = \emptyset \in I$ and $(F - A) = \emptyset \in I$. Consequently, A is an I-pre-closed set.

Corollary 2.29. Let I be an ideal on a locally indiscrete topological space (X, τ) , then the following statements are hold:

- (1) Every singleton in X is I -pre-closed set.
- (2) Every open subset of X is I -pre-closed set.
- (3) Every open subset of X is I -pre-clopen set.
- (4) Every I -semi-open set is I -pre-clopen set.

Theorem 2.30. Let I be an ideal on Extremally disconnected topological space (X, τ) , then the concepts I -semi-open and I - α -open are coincide.

Proof Obvious.

3. CONCLUSIONS

Throughout the current paper, the notations of α (resp., semi, pre, β)-open sets in terms of an ideal I have been introduced. The new concepts were found to be generalization of the usual concepts of α (resp., semi, pre, β)-open sets. Several of their topological properties of these concepts were investigated.

One of them, the concepts of I - α -open set and I -pre-open set are independent; although every α -open set is pre-open set.

If X is the locally indiscrete topological space, then every I - α -open is an I -pre-open. Relationships were established by the new definitions of I -semi-open and I -pre-open, I - α open and I - β -open sets. The new findings in this paper will enhance and promote the further study on generalized open sets in terms of an ideal I to carry out a general framework for their applications in practical life.

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