Types of Generalized Open Sets with Ideal

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ABSTRACT
The aim of this paper is to introduce and study the notions of α (resp., pre, semi, β)-open sets in terms of an ideal I. These concepts generalize the usual notions of α (resp., pre, semi, β) open sets. Also, several of their topological properties are investigated.

Keywords
Pre-open set, Semi-open set, α-open set, β-open set, Topological ideal, I-pre-open set, I-semi-open set.

1. INTRODUCTION
It is well-known that a large number of papers is devoted to study classes of subsets of a topological space, containing the class of open sets, and possessing properties more or less similar to those of open sets e.g. pre-open sets [7, 10, 11], semi-open sets [9], α-open sets [13] and β-open sets [1] (called semi-preopen sets [2]).

Relevant to the present work, the idea of using topological ideals in describing topological notions, which for some years now has been an interesting subject for investigation [see some of the pioneering works in 3, 4, 5]. We recall here that an ideal I on a topological space (X,τ) is a non-empty collection of subsets of X having the heredity property (that is, if A∈I and B⊆A, then B∈I) and also satisfying finite additively (that is, if A, B∈I, then A∪B∈I).

The concept of I-semi-open and I-pre-open sets in ideal topological spaces were introduced and studied by [12] and [14], respectively. These definitions turned our attention to proceed in this direction and introduce the concept of α (resp.,β)-open sets with respect to an ideal I, which is said to be I-α (resp., I-β)-open sets. These notations are extensions of usual concepts α (resp.,β)-open sets. Some of their basic properties will be obtained. These new definitions enable us to establish relationships between definitions of I-semi-open, I-pre-open sets, I-α-open sets and I-β-open sets.

Throughout the current work a topological space (X,τ) (or simply X) will be utilized, where no separation axioms are assumed. The usual notations Cl (A) for the closure and Int (A) for the interior of a subset A of a topological space (X,τ) is used.

We recall some notions which are used in our sequel.
Definition 1.1. A subset A of a topological space (X,τ) is said to be:

(1) Pre-open set [10], if A⊆Int(Cl(A)) and a pre-closed set, if Cl(Int(A))⊆A.

(2) Semi-open set [9], if A⊆Cl(Int(A)) and a semi-closed set, if Int(Cl(A))⊆A.

(3) α-open set [13], if A⊆Int(Cl(Int(A))) and a α-closed set, if Cl(Int(Cl(A)))⊆A.

(4) β-open set [1], if A⊆Cl (Int(Cl(A))) and a β-closed set, if Int(Cl(Int(A)))⊆A.

Definition 1.2. A subset A of a topological space (X,τ) is said to be:
(1) Dense set [8], if Cl (A)=X.

(2) Nowhere dense set [6], if Int Cl (A) =∅.

2. GENERALIZED OPEN SETS WITH RESPECT TO AN IDEAL
In this section we will discuss I-α (resp., I- pre, I-semi, I-β)-open sets in the topological space.

Definition 2.1.A subset A of X is said to be:
(1) α-open with respect to an ideal I (written as I -α-open), if there exists an open set U such that (A – Int Cl (U)) ∈I and (U - A) ∈I.

(2) Pre-open with respect to an ideal I (written as I-pre-open) [14], if there exists an open set U such that (A - U) ∈I and (U – Cl (A)) ∈I.

(3) Semi-open with respect to an ideal I (written as I-semi-open) [12], if there exists an open set U such that (A – Cl (U)) ∈I and (U - A) ∈I.

(4) β-open with respect to an ideal I (written as I-β- open), if there exists an open set U such that (A – Cl (U)) ∈I and (U – Cl (A)) ∈I.

(5) α (resp., pre, semi, β)-closed set with respect to an ideal [briefly, I– α (resp., I–pre, I–semi, I–β)-closed] set iff the complement (X - A) is I-α (resp., I–pre, I–semi, I–β)-open set.

Definition 2.2.If a subset A of X is both I-α (resp., I-pre, I-semi, I-β)-open set and I-α (resp., I-pre, I-semi, I-β)-closed set, then it is called I-α (resp., I-pre, I-semi, I-β)-clopen set.

Lemma 2.3.[12] If $A \in I$, then A is I -semi-open set.

Remark 2.4.[11]

α-open → pre-open
↓ ↓
semi-open → β-open

Remark 2.5. For any ideal I on X and by using Definitions 1.1 and 2.1., the following hold:

α-open	→	I-α-open
semi-open	→	I-semi-open
pre-open	→	I- pre-open
β-open	→	I-β-open

The following examples show that the reverse implications are not satisfied.

Example 2.6 Consider a topological space (X,τ); X= {a, b, c} and τ= {∅, {c}, {a, c}, X}. Choose I= {∅, {b}, {c}, {b, c}} and observe that:

(1) {b} is I-pre (resp., I-semi) -open; however, it is not pre-(resp., semi)-open set.

(2) {a} is I-α (resp., I-β)-open; however, it is not α (resp., β) –open set.

Lemma 2.7.For any ideal I on X, the following hold:

I-α-open I- pre-open
↓ ↓
I-semi-open → I- β-open

Proof In view of Definition 2.1., the proof is obvious.

Example2.8. Consider the topological space (X,τ); in example 2.6 with I= {∅, {c}}. Then, {b} is I-α-open; and is not I-pre-open.

Example 2.9.Consider a topological space (X,τ); X = {a, b, c, d}, τ= {∅, X, {a, b, c}, {a, c, d}, {a, c}} and I= {∅, {d}}. Then:

(1) {a} is I-pre-open; however, it is not I-α-open set.

(2) {a} is I-β-open; however, it is not I- semi–open set.

Example 2.10.Consider a topological space (X,τ); X= {a, b, c}, τ= {∅, {b}, {c}, {b, c}, X} and I= {∅, {c}}. Then:

(1) {a} is I-semi-open; however, it is not I-α-open set.

(2) {a, c} is I-β-open; however, it is not I- pre–open set.

According to examples 2.8 and (i) in 2.9 the concepts of I-α open set and I-pre-open set are independent although every α open set is pre-open set in [11].

Lemma 2.11.(see [4]) If A is a dense subset of (X,τ), then it is pre-open set, moreover it is β-open set.

Corollary 2.12. If A is a dense subset of (X,τ), then it is I-pre-open set and I-β-open set.

Proof In view of Remark 2.5., Lemma 2.7 and Lemma 2.11, the desired proof is clear.

Lemma 2.13. If A is a nowhere dense subset of (X,τ), then it is α-closed set and semi-closed set.

Corollary 2.14 If A is a nowhere dense subset of (X,τ) and $B \subseteq A$. Then,

(1) A is I-α-closed set and I-semi-closed set.

(2) B is I-α-closed set and I-semi-closed set.

(3) Cl(A) is I-α-closed set and I-semi-closed set

Remark 2.15. Notice that τ-open set U is necessarily non-empty, otherwise for any I-α (resp., I-pre, I-β) -open subset A of X and U=∅, then A is I-semi-open set.

In the next part, we discuss some of the properties of I-α (resp., I-pre, I-semi, I-β)-open sets.

Proposition 2.16.[12, 14] If A and B are both I-semi (resp., I-pre)-open set, then so is their union $A \cup B$.

Theorem 2.17.The union of two I-α (resp., I-β)-open sets of (X,τ) is also an I-α (resp., I-β)-open set.

Proof Let A and B be I-α-open sets, then $\exists U, V \in \tau$ such that $(A - \text{Int Cl }(U)) \in I$, $(U - A) \in I$, $(B - \text{Int Cl }(V)) \in I$ and $(V - B) \in I$. Choose $W=U \cup V$, hence $(U-(A \cup B)) \subseteq (U - A) \in I$, $(V-(A \cup B)) \subseteq (V - A) \in I$ and so, $(U-(A \cup B)) \cup (V-(A \cup B)) \in I$. Consequently, $(W - (A \cup B)) \in I$. Also, $(A - \text{Int Cl }(W)) \subseteq (A - \text{Int Cl }(U)) \in I$ and $(B - \text{Int Cl }(W)) \subseteq (B - \text{Int Cl }(V)) \in I$. Then, $(A - \text{Int Cl }(W)) \cup (B - \text{Int Cl }(W)) \in I$ and so $(A \cup B - \text{Int Cl }(W)) \subseteq (A - \text{Int Cl }(U)) \cup (B - \text{Int Cl }(V)) \in I$. Thus $A \cup B$ is I-α-open set. The rest of the proof is similar.

The intersection of two I-α (resp., I-pre, I-semi, I-β)-open sets need not to be an I-α (resp., I-pre, I-semi, I-β)-open set as shown by the following example.

Example 2.18. Consider a topological space (X,τ); X={a, b, c, d} and τ= {∅, {a, b, d}, {a, d}, {a, b}, {b, d}, {d}, {b}, {a}, X}. Choose I= {∅, {c}}.

(1) If A= {a, c}, B= {b, c}, then A and B are I-α-open sets but their intersection $A \cap B$= {c} is not an I-α-open set.

(2) If A= {a, c}, B= {b, c}, then A and B are I-semi-open sets but their intersection $A \cap B$= {c} is not an I-semi-open set.

(3) If A= {a, c}, B= {c, d}, then A and B are I-pre-open sets but their intersection $A \cap B$= {c} is not an I-pre-open set.

(4) If A= {a, c}, B= {b, c}, then A and B are I-β-open sets but their intersection $A \cap B$= {c} is not an I-β-open set.

Theorem 2.19 (1) The intersection of I-pre (resp., I-semi, I-α, I-β)-open and open sets is an I-pre (resp., I-semi, I-α, I-β)-open set.

(2) The intersection of dense and open sets is an I-pre-open set. Moreover, it is an I-β-open set.

Proof (1) Let A be an I-pre-open set and B be an open set, then there exists an open set U such that (A - U)∈I and (U – Cl(A))∈I. Since (U∩B) is an open set, then (A∩B) - (U∩B) = (A∩B) - U ⊆(A - U) ∈I. Since B is open, then Cl (A) ∩B⊆Cl (A∩B) and ((U∩B) – Cl (A)) ∈I. Hence, (U∩B) - Cl (A∩B) ⊆((U∩B) – Cl (A) ∩B) ⊆((U∩B) – Cl (A)) ∈I. Consequently, (A∩B) is an I-pre-open set. We can prove the rest of the theorem similarly.

(2) Let A be a dense set and B be an open set, then (A∩B) - B=∅∈I. Since Cl (A)∩B⊆Cl (A∩B) and Cl (A) =X, then B⊆Cl (A∩B) and so B – Cl (A∩B) =∅∈I.Consequently, (A∩B) is an I-pre-open set. By using Lemma 2.7., (A∩B) is an I-β-open set.

Theorem 2.20 If A is dense and B is open in the topological space (X,τ), then Cl(A∩B) is an I-semi-open set and I-β-open set.

Proof Let A be a dense set and B be an open set, then Cl (A)∩B⊆Cl (A∩B) and Cl (A) =X. Hence B⊆Cl (A∩B) and so B – Cl (A∩B) =∅∈I. Also, Cl (A∩B) - Cl B=∅∈I Consequently, Cl(A∩B) is an I-semi-open set. By using Lemma 2.7., Cl(A∩B) is an I-β-open set.

The proofs of the next theorems are obvious and so are omitted.

Theorem 2.21. Let I= {∅} on a topological space X. Then:
(1) The concepts of α-openness and I-α-openness are the same.

(2) The concepts of pre-openness and I-pre-openness are the same.

(3) The concepts of semi-openness and I-semi-openness are the same.

(4) The concepts of β-openness and I-β-openness are the same.

Theorem 2.22.Let I and J be two ideals on a topological space X. Then we have,
(1) If I⊆J, then every I-α (resp., I-pre, I-semi, I-β)-open set is J-α (resp., J-pre, J-semi, J-β)-open set.

(2) If a subset A is an (I∩J)-α (resp., (I∩J)-pre, (I∩J)-semi, (I∩J)-β)-open set, then it is simultaneously I-α (resp., I-pre, I-semi, I-β)-open set and J-α (resp., J-pre, J-semi, J-β)-open set.

Theorem 2.23.If B is an I-pre (resp., I-β)-open set and A⊆B⊆Cl (A) in (X,τ), then A is an I-pre (resp., I-β)-open set.

Corollary 2.24. Let A be a subset of a topological space X such that Cl (A) is an I-pre (resp., I-β)-open set. Then A is an I-pre (resp., I-β)-open set for any ideal I on X.

Corollary 2.25.Let A and B be subsets of a topological space X such that B is open, A⊆B, and A is dense in B (that is, B⊆Cl (A)). Then A is an I-pre-open set for any ideal I on X.

Definition 2.26.(see [7]) A topological space (X,τ) is said to be:
(1) Hyper connected, if every nonempty open subset of X is dense.

(2) Locally indiscrete, if every open subset of X is closed.

(3) Extremally disconnected, if the closure of every open subset of X is open.

Theorem 2.27. Let I be an ideal on a hyper connected topological space (X,τ), A⊆X and the collection of open subsets of X satisfies the finite intersection property(see [8]). Then the following statements hold:
(1) If A is I-pre-closed set, then A is dense.

(2) If A is I-pre-closed set, then it is I-pre-open set.

(3) If A is I-pre-closed set and A⊆B, then B is I-pre-open set.

(4) If A is I-pre-closed set. Then, A∪B is I-pre-open, for any subset B of X.

(5) If A is I-pre-open and B⊆A. Then, A∩B is I-pre-closed, for any subset B of X.

Proof (1) Let A be an I-pre-closed subset of X, then there exists closed set F such that (Int (A) - F) ∈I and (F - A) ∈I. Since X is hyper connected, the collection of open subsets of X satisfies the finite intersection property and (Int (A) - F) is non-empty open set, then (Int (A) - F) is dense and X= Cl (Int (A) - F) ⊆Cl (A) and so A is dense.

(2) In view of Lemma 2.7 and by using (1) of this theorem. Since A is an I-pre-closed set, then it is I-pre-open.

(3) Let A be I-pre-closed and A⊆B and by using (1) of this theorem, then A is dense and so B is dense. In view of Lemma 2.7 B is I-pre-open set. The rest of the proofs are obvious.

Theorem 2.28. Let I be an ideal on a locally indiscrete topological space (X,τ), then the following statements are hold:
(1) Every I-semi-open is an I-pre-open.

(2) Every I-semi-open is an I-α-open.

(3) Every I-β-open is an I-pre-open.

(4) Every I-α-open is an I-pre-open.

(5) If A⊆X, then A is an I-pre-closed.

Proof We shall prove (1), (5) only and the rest of the proof is obvious.
(1) Let A be an I-semi-open, then there exists an open set U such that (A –Cl (U))∈I and (U - A) ∈I. Since X is locally indiscrete, then A is an I-pre-open set.

(4) Let A⊆X. Choose F=Int (A). Since X is locally indiscrete, then F is closed, (Int (A) -F) = ∅ ∈I and (F - A) = ∅∈I. Consequently, A is an I-pre-closed set.

Corollary 2.29. Let I be an ideal on a locally indiscrete topological space (X,τ), then the following statements are hold:

(1) Every singleton in X is I-pre-closed set.

(2) Every open subset of X is I-pre-closed set.

(3) Every open subset of X is I-pre-clopen set.

(4) Every I-semi-open set is I-pre-clopen set.

Theorem 2.30.Let I be an ideal on Extremally disconnected topological space (X,τ), then the concepts I-semi-open and I-α-open are coincide.

Proof Obvious.

3. CONCLUSIONS

Throughout the current paper, the notations of α (resp., semi, pre,β)-open sets in terms of an ideal I have been introduced. The new concepts were found to be generalization of the usual concepts of α(resp., semi, pre,β)-open sets. Several of their topological properties of these concepts were investigated.

One of them, the concepts of I-α-open set and I-pre-open set are independent; although every α-open set is pre-open set.

If X is the locally indiscrete topological space, then every I-α-open is an I-pre-open. Relationships were established by the new definitions of I-semi-open and I- pre-open, I-α open and I-β-open sets. The new findings in this paper will enhance and promote the further study on generalized open sets in terms of an ideal I to carry out a general framework for their applications in practical life.

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