Control of Lotka-Volterra three species system via a high gain observer design

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ABSTRACT

This paper presents the role of the nonlinear observer to estimate the state variable for the continous lotka-volterra system. A theoretical frame is provided in order to show the convergence characteristics of the proposed observer under some satisfied assumptions. The proposed methodology is applied to a class of Lotka-Volterra systems with three species. Numerical simulations are shown to validate and demonstrate the effectiveness of the proposed high gain observer.

Keywords

Dynamic System, Lotka-Volterra Model, Hight Gain, Observer Design.

1. INTRODUCTION

Understanding the behaviour of the interaction between the species may help scientists to prevent those events from happening. The real interaction of prey-predator in nature is complex and comprises both interspecies and external environmental factors. Therefore, several simplifications are usually assumed so that a basic model can be constructed and then developed or modified to approach the real system [1].
Prey-Predator systems is a chaotic nonlinear dynamical systems, which is highly sensitive to initial conditions. There sensitive nature is usually called as the butterfly effect.
In 1963, Lorenz first observed the chaos phenomenon in weather models. Since then, a large number of chaos phenomena and chaos behaviour have been discovered in physical, social, economical, biological and electrical systems [5]. Knowing the state system is necessary to solve many control theory problems, for example, stabilizing a system using state feedback [7].
Most of the modern control design methods, especially for nonlinear systems, use a state feedback as the controller called an observer [2].
The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances [9,3,10]. As we know, it is almost impossible to measure all the elements of the state vector in practice (e.g., the unknown state variables, fault signals, etc.).
The idea of observation suggests that we calculate inner quantities x(t) by the knowledge of the essential parts of dynamic process as well as the measurable input quantities u(t) and may be the output quantities y(t) as well. This is done by the observer, which delivers an estimate of the inner quantities xobs(t) [7].
The high gain observer is an appropriate technique for several class of nonlinear systems. The basic idea of this approach is to dominate the nonlinear behavior of the system by applying high gains [6].
The main contribution in this work is to investigate the high gain observer of Lotka-Volterra system with three species.This observer will be as a numerical technique to reconstruct unknown variables is designed.. Here arises a basic practical question:
- Would it be possible to reconstruct the unknown signals?
We give an answer to this question by introducing a basic definition called Algebraic Observability property (AOP) [4].
The rest of this paper is organized as follows. We first describes the problematic of the continuous model, under some satisfied assumptions. Next we give an observer design that we will appliying for our lotka-volterra model. Then we investigate the technique for the estimation of the lotka-volterra model in an invariant domain. Finally a numerical example is given and simulation results are shown.

2. THE PROBLEM FORMULATION

As is it well known, the Lotka-Volterra model describes interactions between several species in an ecosystem, predators and preys [5], we considered three equations, one describes the variations of the predator population, the two others equations describe the prey population.

2.1 Model of Three Equations

In this paper, by modifying the classical Lotka-Volterra model, we analyse and simulate the dynamics of a threespecies. With non-dimensionalisation, the system of three species can be written as:

$$\begin{cases} \dot{x}_1 = x_1(a_1 - b_1x_1 - c_1x_2) \\ \dot{x}_2 = x_2(-a_2 + b_2x_1 + c_2x_3) \\ \dot{x}_3 = x_3(a_3 - b_3x_2 - c_3x_3) \end{cases} \quad (1)$$

Where:
x_1,x_3 are prey populations,
x_2 is the predator population;
$a_i, b_i, c_i > 0$ for all i=1,2,3.
Moreover, $y = x_1$ is the system measured output.

Since populations are nonnegative, we will restrict our attention to the nonnegative orthants;
$\{(x_1, x_2, x_3)|x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\} \in R^3$ and the positive orthants $\{(x_1, x_2, x_3)|x_1 > 0, x_2 > 0, x_3 > 0\} \in R^3$.
It is worthwhile to mention that the nonnegative and the positive quadrants are positively invariant for the general Lotka-Volterra model.

2.2 Equilibrium Point Analysis

According to Hilborn (1994) and May (2001), the equilibrium points of (1) denoted by , are the zeros of its nonlinear algebraic system which can be written as [12]:

$$\begin{cases} x_1(a_1 - b_1x_1 - c_1x_2) = 0 \\ x_2(-a_2 + b_2x_1 + c_2x_3) = 0 \\ x_3(a_3 - b_3x_2 - c_3x_3) = 0 \end{cases} \quad (2)$$

By considering the positivity of the parameters and the unknowns, we have the positive equilibrium points given by:

$$\tilde{x}_1 = \frac{a_2c_1c_3 + a_1b_3c_2 - a_3c_1c_2}{b_1b_3c_2 + b_2c_1c_3},$$
$$\tilde{x}_2 = \frac{a_1b_2c_3 - a_2b_1c_3 + a_3b_1c_2}{b_1b_3c_2 + b_2c_1c_3},$$
$$\tilde{x}_3 = \frac{a_2b_1b_3 + a_3b_2c_1 - a_1b_2b_3}{b_1b_3c_2 + b_2c_1c_3}.$$

3. HIGH OBSERVER DESIGN

State observers (software sensors) are able to provide a continuous estimation of some signals which are not measured by hardware sensors. They need a mathematical model of the process and hardware measurements of some other signals. An observer is a dynamic system whose input includes the control u and the output y and whose output is an estimate of the state vector $\widehat{x}$.

Now, consider a general representation of the Lotka-volterra models, which can be generally bye the following lumped parameter model:

$$\begin{cases} \dot{x} &= Ax + F(x) \\ y &= h(x) = Cx \end{cases} \quad (3)$$

Where:
C = (1 0 0),
$x \in R^n$ is the vector of states, $x = (x_1, ..., x_n)^T$,
$F : R^n \longrightarrow R^n$ is a nonlinear vector field,

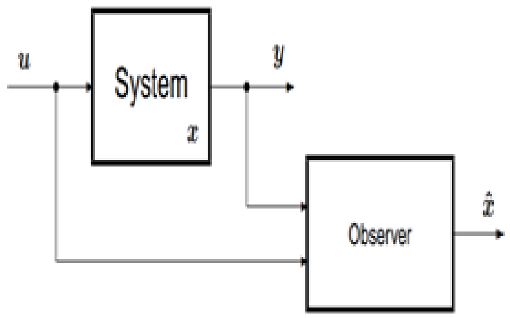

Fig 1: **Principle of the observer**

$y \in R^m$ is the system measured output, with $m < n$, and

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ : & 1 & 0 & 0 & 0 \\ 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is necessary the design of an auxiliary system so called observer system to reconstruct then known states or unmeasurables. Firstly, we give necessary and sufficient conditions to establish whether the system (3) is observable.
Now, consider the following assumptions [5]:

A1. The system given in Eqs. (1) is locally uniformly observable (Gauthier et al., 1992), hence for all $x \in R^n$, satisfies the observability rank condition:

$$rang \frac{\partial}{\partial x} \nu = n \quad (4)$$

Here ϑ is the observability vector function defined:
$\vartheta = (dL_f^0h, dL_f^1h, \ldots, dL_f^{n-1}h)^T L_f^r h$ the r-order Lie derivatives, which are the directional derivatives of the corresponding state variables along the measured output trajectory. And $dL_f^r h$ are the differentials of the rth-order Lie derivatives defined recursively as follows:

$$L_f^0 h := h,$$
$$dL_f^0 h := dh = (\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}).$$
$$L_f^1 h :=< dh, f >= \textstyle\sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i,$$
$$dL_f^1 h := (\frac{\partial}{\partial x_1}(\textstyle\sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i), \ldots, \frac{\partial}{\partial x_1}(\textstyle\sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i)).$$
$$L_f^r h :=< dL_f^{r-1} h, f >= L_f(L_f^{r-1} h), r \succeq 2.$$

A2. All the trajectories $x(t, x_0)$, $x_0 \in R^n$ of the system (1) are bounded. Considering the set $\Omega \in R^n$ as the corresponding physically realizable domain, such that: $\Omega = \{(x_i)_{i=1}^n \in R_+^n / 0 \leq x_i \leq x_{max}\}$
In most practical cases,Ω will be an open connected relatively compact subset of R^n, and in the ideal cases, will be positively invariant under the dynamic (3).

In order to analyze the estimation error $\varepsilon = x - \widehat{x}$ we consider the next assumption.

A3. The nonlinear difference vector function $\Delta F = F(x) - F(\widehat{x})$ is Lipschitz bounded i.e $\Delta F < \Lambda|\varepsilon|$
Condition A3 can be fulfilled satisfied if the following supremumis finite:

$$\Lambda := sup_{x\in\Omega}\|F'(x)\|$$

Where $F'(x)$ is the Jacobian.

4. PROPOSITIONS

4.1 Proposition 1

Consider that system (3) satisfies A1, A2 and A3 (with these assumptions is possible to construct an observer)then, the system (5) is an asymptotic observer for the system (3):

$$\dot{\widehat{x}} = A\widehat{x} + F(\widehat{x}) - \theta d_\theta^{-1} S_1^{-1} C'(C\widehat{x} - y) \tag{5}$$

As will be shown in Section 2.2, for system (2) , is not difcult to provide a simple algebraic condition for the existence of an equilibrium in mathematical sense, however its positivity depends on the model parameters, that means, for any set of initial conditions the trajectories for $t \geq 0$ tend or cycle around this equilibrium point in the positive quadrant, in this form, condition A2 is satisfied.
The three-dimensional predatorprey system (1), the observability matrix is given by:

$$\partial\vartheta = \begin{pmatrix} 1 & 0 & 0 \\ a_1 - 2b_1x_1 - c_1x_2 & -c_1x_1 & 0 \\ \dfrac{\partial\vartheta_3}{\partial x_1} & \dfrac{\partial\vartheta_3}{\partial x_2} & -c_1c_2x_1x2 \end{pmatrix}$$

where:

$$\frac{\partial\vartheta_3}{\partial x_1} = (a_1 - 2b_1x_1 - c_1x_2)^2 - 2b_1x_1(a_1 - b_1x_1 - c_1x_2) \\ -c_1x_2(-a_2 + 2b_2x_1 + c_2x_3)$$

$$\frac{\partial\vartheta_3}{\partial x_2} = -c_1x_1(2a_1 - a_2 + (b_2 - 3b_1)x_1 - 2c_1x_2 + c_2x_3)$$

Then:

$$\det(\frac{\partial\vartheta}{\partial x}) = c_1^2c_2x_1^2x_2$$

If populations x_1, x_2 are nonzero, then the observability rank condition is fullled. By using assumption A1 the three-species model (1) is locally uniformly observable, in this sense, all state variables can be estimated if we only observe the prey population x_1.

Notice also that, if we take the initial condition such that $x_1(0) = 0$, then for all $t \geq 0$, the output is zero. In these systems, preys populations cannot grow if they are not present at the beginning of the story.

4.2 Proposition 2

For any initial condition $x(0) \in \Omega$ and any $\widehat{x}(0) \in \Omega$ and for θ large enough the system (1) satisfying assumptions (1)(2)(3), can be estimated by the following dynamical system [8,11]:

$$\dot{\widehat{x}} = A\widehat{x} + F(\widehat{x}) - \theta d_\theta^{-1} S_1^{-1} C'(C\widehat{x} - y) \tag{6}$$

Where S_1 is the symetric positive definite solution of the algebraic equation:

$$\theta S_\theta + A'S_\theta + S_\theta A - C'C = 0$$

For $\theta = 1$ and it can be expressed as:

$S_1(i,j) = (-1)^{i+j}C_{i+j-2}^{j-1}$, for $1 \leq i, j \leq n$, where
$C_j^i = \dfrac{j!}{i!(j-i)!}$
d_θ is a diagonal matrix defined by :

$$d_\theta = diag(1, \frac{1}{\theta}, \ldots, \frac{1}{\theta^n})$$

4.3 Lemma

For θ large enough the system below is an observer for the system:

$$\dot{\widehat{x}} = A\widehat{x} + F(\widehat{x}) - \theta d_\theta^{-1} S_1^{-1} C'(C\widehat{x} - y) \tag{7}$$

Proof
Let $e = \widehat{x} - x$
Then on can check that:

$$\dot{e} = (A - \theta d_\theta^{-1} S_1^{-1} C'C)e + \Delta(F)$$

Where:
$\Delta(F) = F(\widehat{x}) - F(x)$

F is lipschitz with the constant Λ so:

$$\|\Delta F\| \leq \Lambda\|e\|$$

Let V_θ a candidate lyapunov equation for the system (1):

$$V_\theta = \frac{1}{\theta}e'd_\theta S_1 d_\theta e$$

The time derivative of V_θ computed along solution of the differential equations (1) is given by:

$$\dot{V_\theta} = \frac{1}{\theta}e'd_\theta(d_\theta^{-1}A'd_\theta S_1 + S_1 d_\theta A d_\theta^{-1} - \theta d_\theta^{-1}C'C - \theta C'C d_\theta^{-1})d_\theta e \\ +2\frac{1}{\theta}(\Delta(F))d_\theta S_1 d_\theta e$$

Taking into account the algebraic equation:

$$\theta S_\theta + A'S_\theta + S_\theta A - C'C = 0$$

And

$$d_\theta A d_\theta^{-1} = \theta A, C'Cd_\theta^{-1} = C'C$$

It follows:

$$\dot{V}_\theta = \frac{1}{\theta}e'd_\theta(-\theta S_1 - \theta C'C)d_\theta e + +2\frac{1}{\theta}(\Delta(F))d_\theta S_1 d_\theta e \quad (8)$$

Using the above inequality:

$$\lambda_{min}(S_1)\|d_\theta e\|^2 \leq e'd_\theta S_1 d_\theta e \leq \lambda_{max}(S_1)\|d_\theta e\|^2$$

Where $\lambda_{min}(S_1)$ and $\lambda_{max}(S_1)$ are respectively the minimum and the maximum eigenvalues of S_1.

5. NUMERICAL EXAMPLES

Below, we expose a system that we claim to be an observer for [8].

$$\dot{\widehat{x}} = A\widehat{x} + F(\widehat{x}) - \theta d_\theta^{-1} S_1^{-1} C'(C\widehat{x} - y)$$

Where:

$C = (100), C' = (100)^T$, A= $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

F($\widehat{x}$)= $\begin{bmatrix} a_1x_1 - b_1x_1^2 - c_1x_1x_2 - x_2 \\ -a_2x_2 + b_2x_1x_2 + c_2x_2x_3 - x_3 \\ a_3x_3 - b_3x_2x_3 - c_3x_3^2 \end{bmatrix}$,

S_1= $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ 1 & -3 & 6 \end{pmatrix}$.

We have:

det(S_1)=1 $\neq 0$

Then:

$$S_1^{-1} = \frac{1}{det(S_1)} * \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

d_θ= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\theta} & 0 \\ 0 & 0 & \frac{1}{\theta^2} \end{pmatrix}$, d_θ^{-1}= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix}$.

Calculate the terms of

$$\theta * d_\theta^{-1} * S_1^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta \\ 3\theta^2 & 2\theta^2 & 2\theta^2 \\ \theta^3 & 2\theta^3 & \theta^3 \end{pmatrix},$$

$$C' * C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C' * C(\widehat{x} - x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 - x_1 \\ \widehat{x}_2 - x_2 \\ \widehat{x}_3 - x_3 \end{bmatrix} = \begin{bmatrix} \widehat{x}_1 - x_1 \\ 0 \\ 0 \end{bmatrix},$$

$$\theta * d_\theta^{-1} * S_1^{-1} * C' * C(\widehat{x} - x) = \begin{bmatrix} 3\theta(\widehat{x}_1 - x_1) \\ 3\theta^2(\widehat{x}_1 - x_1) \\ \theta^3(\widehat{x}_1 - x_1) \end{bmatrix}.$$

6. RESULTS AND DISCUSSION

In order to measure the performance of the proposed observer under different polynomial degrees is employing the measure the impact of the error, suggested in Ogunnaike and Ray (1994), is the Integral Time-Weighted Squared Error (ITSE) dened in [5].
ITSE exhibits the advantage of heavy penalization of large errors at long time; therefore is a good measure of resilience of the observer.

$$ITSE = \int_0^\infty t.\varepsilon^2 \quad (9)$$

With the observation error as $\varepsilon = x - \widehat{x}$.
Firstly, we present some simulations for Lotka-Volterra model given in (1) and its corresponding observer (5).
We have taken the parameters values as:

$a_1 = 3, a_2 = 2, a_3 = 6,$
$b_1 = 0.01, b_2 = 0.01, b_3 = 0.02,$
$c_1 = 0.01, c_2 = 0.02, c_3 = 0.02.$

And the initial conditions:

$x_1 = 10, x_2 = 20, x_3 = 5.$
$\widehat{x}_1 = 80, \widehat{x}_2 = 90, \widehat{x}_3 = 100.$

6.1 Figures

In Figs. 2 , 3 and 4 are shown the state variables and their corresponding estimated states, using $\theta = 2, \theta = 5$ and $\theta = 15$ in system (5).
The three values of guarantees asymptotic convergence, especially for the third one.

Any initial state $0 < x_1(0) \neq \tilde{x}_1, 0 < x_2(0) \neq \tilde{x}_2, 0 < x_3(0) \neq \tilde{x}_3$ leads to this equilibrium point in the positive quadrant.
Furthermore, in Figs. 5, 6, 7, is presented the performance index (ITSE) given in (9), for three order predator-prey model (1).
We have taken $\theta = 2, 5, 15$, for the observer (5). It should be noted that the value of the corresponding performance index decrease as θ increase.
Fig. 8 shows that the model has no periodic fluctuations. And by algebraic manipulations an asymptotically stable equilibrium point of (1) is obtained as:

$$\tilde{x}_1 = \frac{a_2c_1c_3 + a_1b_3c_2 - a_3c_1c_2}{b_1b_3c_2 + b_2c_1c_3} = 66.90,$$

$$\tilde{x}_2 = \frac{a_1b_2c_3 - a_2b_1c_3 + a_3b_1c_2}{b_1b_3c_2 + b_2c_1c_3} = 200.04,$$

$$\tilde{x}_3 = \frac{a_2b_1b_3 + a_3b_2c_1 - a_1b_2b_3}{b_1b_3c_2 + b_2c_1c_3} = 66.72.$$

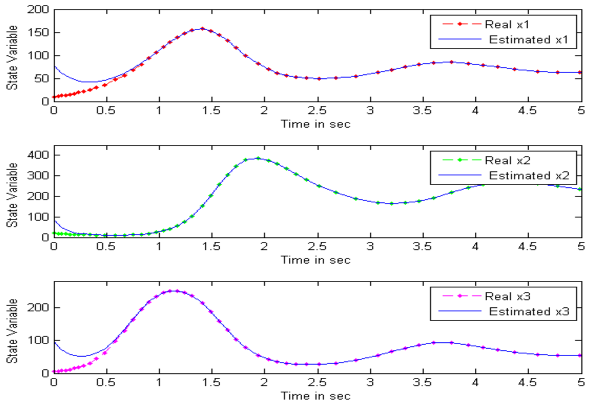

Fig 2: Convergence asymptotic of the observer with $\theta = 2$

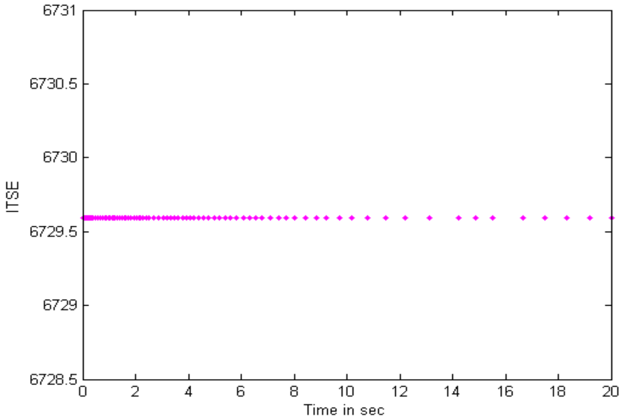

Fig 5: Performance index for predator-prey model for θ=2

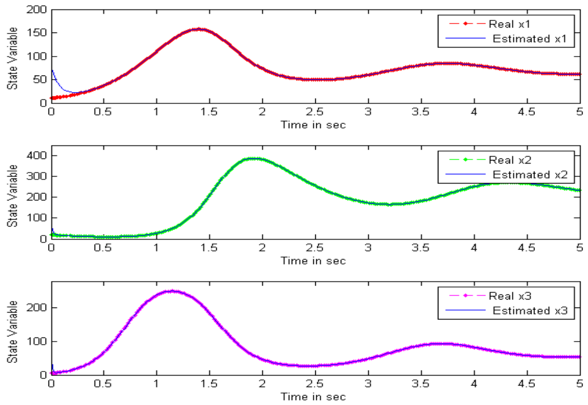

Fig 3: Convergence asymptotic of the observer with $\theta = 5$

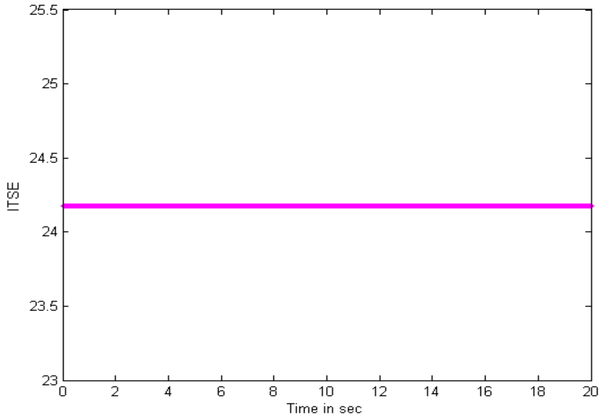

Fig 6: Performance index for predator-prey model for θ=5

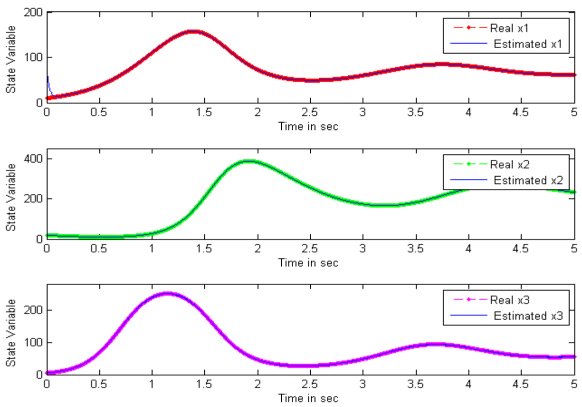

Fig 4: Convergence asymptotic of the observer with $\theta = 15$

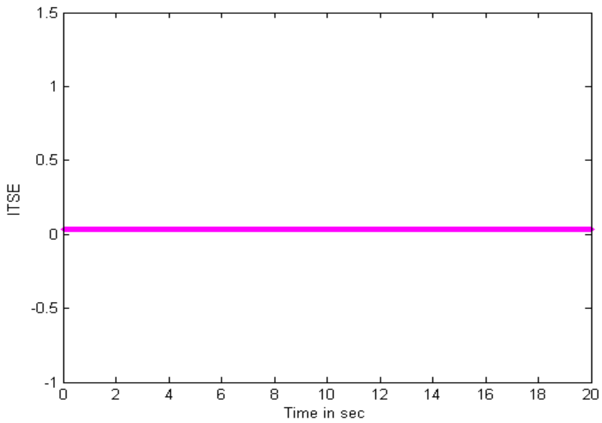

Fig 7: Performance index for predator-prey model for θ=15

Table 1. Corresponding results of the ITSE index for each simulated value of θ

Variable	Description	Value	ITSE Value
Theta	Observer Gain	2	6729,6
		5	24,18
		15	0,032

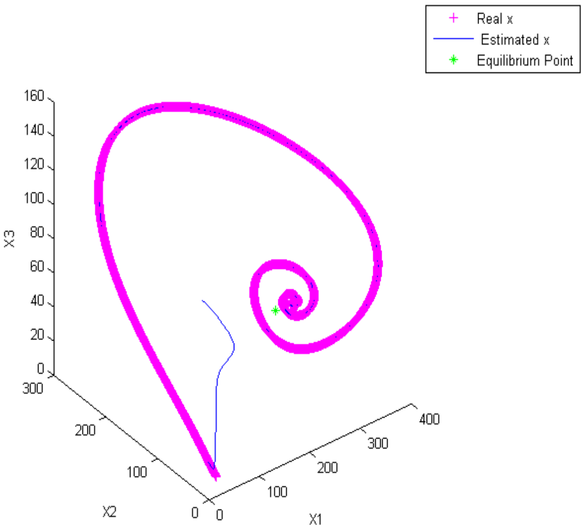

Fig 8: Three order predator-prey model trajectories with equilibrium point

The table 1 shows the estimated values of the corresponding performance index that decrease as θ increase.

7. CONCLUSION

In this work we have presented a high order observer to solve the problem of control in predatorprey systems.
The convergence of the estimation error is proved under certain conditions and the gain of the observer is explicitly formulated.
The proposed methodology was applied to a class of LotkaVolterra model with three species with success.

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