Associative Binary Operations on a Set with Five Elements

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ABSTRACT

Associativity of a binary operation is one of most fundamental property in algebra. The number of binary operation on a set of only five elements is as large as 298,023,223,876,953,125. In this paper, the authors give the answer of the question "How many binary operations on a five elements are associative?" The main goal of this paper is to extend [3] and [5] for five element set.

Keywords

binary operations, associativity, isomorphism, order, Partition

1. INTRODUCTION

The number of binary operation on a set of only five elements is as large as 298,023,223,876,953,125. To prove this is an easy calculation; there are five different answers for each of the 25 seats of a 5X5 operation table so the number of distinct binary operations is 5^{25} .This calculation is straightforward, but no easy calculation and no educated guess, seems to give the answer to following question:-

"How many binary operations on a five elements are associative?"

The objective of this paper is to answer this question. In other words, how many of the 298,023,223,876,953,125 different binary operations on five-element set are associative. The main goal of this paper is to extend [3] and [5] for five element set.

2. GENERAL CONCEPTS

A *binary operation* (hereafter referred to only as an operation) on a set S is a rule that assigns to each ordered pair (a, b), where a and b are elements of S, exactly one element, denoted by ab, in S.

An operation on a set is *associative* if x (yz) = (xy) z for every x, y and z in S.

An *isomorphism* between S and S' is a one-to-one function Φ mapping S onto S' such that Φ(xy)= Φ(x)Φ(y) for all x and y in S. If there exists an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted by S≈S'.

Order of an element is cardinality of set generated by that element.

3. USEFUL THEOREM

The following theorem about sets, S and S', closed under operations are well known and easy to prove:-

Theorem: - If there exists an isomorphism between S and S' and the operation on S is associative then an operation on S' is also associative.

4. OPERATIONS ON A SET WITH 2, 3 AND 4 ELEMENTS

Number of binary operation on a set of two elements is 16.

Number of associative binary operation on a set of two elements is 8.See [2].

Number of binary operation on a set of three elements is 19683.

Number of associative binary operation on a set of three elements is 113.See [2] &[3].

Number of binary operation on a set of four elements is 4,294,967,296.

Number of associative binary operation on a set of four elements is 3492.See [5]

5. OPERATIONS ON A SET WITH FIVE ELEMENTS

As mentioned in the introduction, the number of possible binary operations on a set of five elements is 298,023,223,876,953,125. We now proceed to answer the question: How many binary operations on a set of five elements are associative?

For a five-element set S proving associatively for a given operation amounts to verify different equations (xy) z=x (yz), where x, y, z are elements of S. A single counterexample suffices to show that a given operation is not associative. Clearly, counterexamples need not to be unique.

Now we discuss alternative ways to check the associativity. Define Φ_a from (S, *) into (S,*), by $\Phi_a(x)$ =a*x for all x in S, where a belong to S. If composition of Φ_a with Φ_b is equal to Φ_{a*b} for all a, b in S then we say that S is associatively under the binary operation *.

Explanation: -

$a*(b*c) = \Phi_a (b*c) = \Phi_a (\Phi_b(c))$ …..(1)

$(a*b)*c = \Phi_{a*b}(c)$ …..(2)

From (1) and (2) we conclude that above definition is equivalent with definition of associatively of binary operation * on set S. See [3]

6. ALOGRITHM FOR FINDING NUMBER OF ASSOCIATIVE BINARY OPERATIONON N-ELEMENT

Algorithm given below is taken from [3], [4] and [5] .The analysis of the associative binary operations on n-element set S will now divide into 3 steps:-

1). Partition the set of n^n mappings in such a way that element of same partition can be obtained by using one-one and onto mapping from S onto S.

2). Rearrange the partition according to their order of any element of the partition.(Say order as k).

3).Calculate the contribution towards number of associative binary operations when one row is fill by the first element of i^{th} partition which can also fill k-1 more rows and remaining row n-k can be filled by i^{th} and onwards partitions (if any) with two conditions given below.

Conditions: - Before starting calculation, firstly we insured that no associative table counted twice. For this, we make some rules:-

(i) If we fixed r^{th} row from any element i^{th} partition (which has order k) then we can fill at least k-1 more rows. Remaining unfilled rows can by filled with element of i^{th} and onwards partition (if any) however selected element of i^{th} partition cannot fill the unfilled rows before selected row.

(ii) If table contains n different entry of i^{th} partition then contribution towards number of associative operations counted is 1/n.

Explanation (How partition reduces the checking of associative work):- Firstly, we prove that Table-1 and Table-2 given below are isomorphic.

$*_1$	a	b	c	d	e
a	e	a	a	c	b
b					
c					
d					
e					

Table-1

$*_2$	a	b	c	d	e
a					
b	b	d	b	c	a
c					
d					
e					

Table-2

Here, if we define an isomorphic mapping from S to S by f (a) =b, f (b) =c, f(c) =a, f(d)=e and f(e)=d. Then we get

$a *_1 a = e$ => $f(e)=f(a) *_2 f(a)$ => $d=b *_2 b$

$a *_1 b = a$ => $f(a)=f(a) *_2 f(b)$ => $b=b *_2 c$

$a *_1 c = a$ => $f(a)=f(a) *_2 f(c)$ => $b=b *_2 a$

$a *_1 d = c$=> $f(c)=f(a) *_2 f(d)$ => $a=b *_2 e$

$a *_1 e = b$ => $f(b)=f(a) *_2 f(e)$ => $c=b *_2 d$

Then composition table after applying above isomorphic mapping to Table-1, we get table-2. Hence, table-1 and table-2 are isomorphic.

Similarly, we can prove that if eaacb comes in second row is isomorphic to bdbca in third row, if eaacb comes in third row is isomorphic to bdbca in first row , if eaacb comes in fourth row is isomorphic to bdbca in fifth row and if eaacb comes in fifth row is isomorphic to bdbca in fouth row. Hence, table with at least one row eaacb is isomorphic to table with at least one row bdbca (row number of bdbca may be different from row number of eaacb).

Hence, there is no need to check associative composition table for bdbca when associative composition table eaacb is checked because they give isomorphic composition table. (By theorem stated in section 3 if a composition table with eaacb is associative then isomorphic composition having row bdbca is also associative).

As above discussion, if we apply any one-one and onto to eaacb we get {bcaad, bcabd, ……., eebad } which forms a partition and now our calculation is reduced 120 times (out of 120 elements now we have to check associativity only for single element).

On the basis of above algorithm, now we proceed towards finding number of associative operations on a set with five elements.

6.1 Step 1^{st}

If set S has n elements, then total number of mapping possible from set S to S= n^n

In our problem n=5, then total number of mapping possible from set S to S= 3125.

Here we consider S={0,1,2,3,4} and first digit of each mapping comes from 0, second comes from 1, third comes from 2, fourth comes from 3 and last comes from 4.

Partition No	First element of Partition	Total number of element in partition
1	00000	5
2	00001	60
3	00004	20
4	00011	60
5	00012	60
6	00013	120
7	00014	120
8	00033	60
9	00034	30
10	00043	30
11	00111	20
12	00112	120

13	00114	60
14	00122	60
15	00123	120
16	00124	120
17	00133	120
18	00134	60
19	00143	60
20	00224	60
21	00234	20
22	00243	60
23	00322	120
24	00342	40
25	01234	1
26	01243	10
27	01322	60
28	01342	20
29	02111	60
30	02112	60
31	02113	120
32	02143	15
33	02311	120
34	02341	30
35	10000	20
36	10001	60
37	10002	120
38	10012	120
39	10022	60
40	10023	120
41	10043	60
42	10342	20
43	12000	60
44	12001	120
45	12003	120
46	12300	120
47	12340	24

6.2 Step 2nd

Partition No	First element of Partition	Order of an element of partition	Generated elements belongs to partition
1	10342	6	1,21,40,21,1,47
2	12340	5	2,2,2,2,47
3	00123	4	3,27,25,46
4	10023	4	4,23,34,41
5	12003	4	5,15,42,15
6	12300	4	6,35,6,45
7	02341	4	7,39,7,47
8	00013	3	8,25,46
9	00112	3	9,25,46
10	00124	3	10,22,44
11	02113	3	11,42,33
12	02311	3	12,12,45
13	10002	3	13,41,34
14	10012	3	14,41,34
15	12001	3	15,15,42
16	00122	3	16,26,46
17	00143	3	17,43,36
18	10022	3	18,41,34
19	12000	3	19,19,43
20	00342	3	20,20,45
21	01342	3	21,21,47
22	00014	2	22,44
23	00133	2	23,41
24	00322	2	24,42
25	00001	2	25,46
26	00011	2	26,46
27	00012	2	27,46
28	00114	2	28,44
29	00134	2	29,43
30	00243	2	30,45
31	01322	2	31,45
32	02111	2	32,43
33	02112	2	33,42
34	10001	2	34,41
35	10043	2	35,45
36	00043	2	36,43
37	00111	2	37,46
38	10000	2	38,44
39	02143	2	39,47
40	01243	2	40,47
41	00033	1	41

42	00224	1	42
43	00034	1	43
44	00004	1	44
45	00234	1	45
46	00000	1	46
47	01234	1	47

6.3 Step 3rd

Partition-No	First element of Partition	Assumed row	Contribution towards number of associative
1	10342	1	0
	10342	2	0
	10342	3	0
	10342	4	0
	10342	5	0
2	12340	1	6
	12340	2	6
	12340	3	6
	12340	4	6
	12340	5	6
3	00123	1	0
	00123	2	0
	00123	3	0
	00123	4	120
	00123	5	120
4	10023	1	0
	10023	2	0
	10023	3	0
	10023	4	120
	10023	5	120
5	12003	1	0
	12003	2	0
	12003	3	0
	12003	4	240
	12003	5	120
6	12300	1	120
	12300	2	60
	12300	3	60
	12300	4	60
	12300	5	0
7	02341	1	0
	02341	2	15
	02341	3	15
	02341	4	15
	02341	5	15
8	00013	1	0
	00013	2	0
	00013	3	0
	00013	4	1620
	00013	5	720
9	00112	1	0
	00112	2	0
	00112	3	780
	00112	4	120
	00112	5	480
10	00124	1	0
	00124	2	0
	00124	3	480
	00124	4	360
	00124	5	0
11	02113	1	0
	02113	2	0
	02113	3	0
	02113	4	240
	02113	5	120
12	02311	1	0
	02311	2	120
	02311	3	60
	02311	4	60
	02311	5	0
13	10002	1	0
	10002	2	0
	10002	3	1200
	10002	4	240
	10002	5	360
14	10012	1	0
	10012	2	0
	10012	3	1680
	10012	4	420
	10012	5	300
15	12001	1	180

	12001	2	180
	12001	3	60
	12001	4	0
	12001	5	0
16	00122	1	0
	00122	2	0
	00122	3	480
	00122	4	60
	00122	5	0
17	00143	1	0
	00143	2	0
	00143	3	0
	00143	4	0
	00143	5	0
18	10022	1	0
	10022	2	0
	10022	3	1080
	10022	4	120
	10022	5	60
19	12000	1	600
	12000	2	30
	12000	3	30
	12000	4	0
	12000	5	0
20	00342	1	0
	00342	2	0
	00342	3	80
	00342	4	80
	00342	5	80
21	01342	1	0
	01342	2	0
	01342	3	120
	01342	4	120
	01342	5	120
22	00014	1	0
	00014	2	9600
	00014	3	1380
	00014	4	1500
	00014	5	0
23	00133	1	0
	00133	2	5520
	00133	3	960
	00133	4	0
	00133	5	1200
24	00322	1	0
	00322	2	0
	00322	3	1680
	00322	4	480
	00322	5	0
25	00001	1	0
	00001	2	29680
	00001	3	4690
	00001	4	4210
	00001	5	4720
26	00011	1	0
	00011	2	15600
	00011	3	1300
	00011	4	1270
	00011	5	970
27	00012	1	0
	00012	2	660
	00012	3	540
	00012	4	210
	00012	5	150
28	00114	1	0
	00114	2	4680
	00114	3	360
	00114	4	180
	00114	5	0
29	00134	1	0
	00134	2	3480
	00134	3	1140
	00134	4	0
	00134	5	0
30	00243	1	0
	00243	2	0
	00243	3	0
	00243	4	1560
	00243	5	1560
31	01322	1	0
	01322	2	0
	01322	3	1680

	01322	4	780
	01322	5	0
32	02111	1	0
	02111	2	840
	02111	3	60
	02111	4	0
	02111	5	0
33	02112	1	0
	02112	2	270
	02112	3	270
	02112	4	0
	02112	5	0
34	10001	1	2460
	10001	2	780
	10001	3	0
	10001	4	0
	10001	5	0
35	10043	1	460
	10043	2	200
	10043	3	0
	10043	4	200
	10043	5	200
36	00043	1	0
	00043	2	0
	00043	3	0
	00043	4	720
	00043	5	720
37	00111	1	0
	00111	2	4160
	00111	3	80
	00111	4	40
	00111	5	20
38	10000	1	3440
	10000	2	20
	10000	3	0
	10000	4	0
	10000	5	0
39	02143	1	0
	02143	2	80
	02143	3	80
	02143	4	50

	02143	5	50
40	01243	1	0
	01243	2	0
	01243	3	0
	01243	4	1190
	01243	5	1190
41	00033	1	12810
	00033	2	0
	00033	3	0
	00033	4	8735
	00033	5	0
42	00224	1	5730
	00033	2	0
	00033	3	4290
	00033	4	0
	00033	5	1470
43	00034	1	5870
	00034	2	0
	00034	3	4290
	00034	4	0
	00034	5	1470
44	00004	1	3205
	00004	2	0
	00004	3	0
	00004	4	0
	00004	5	1425
45	00234	1	1585
	00234	2	0
	00234	3	680
	00234	4	530
	00234	5	420
46	00000	1	536
	00000	2	0
	00000	3	0
	00000	4	0
	00000	5	0
47	01234	1	1
	01234	2	0
	01234	3	0
	01234	4	0
	01234	5	0

Summary of Section

Partition-No	Total contribution towards number of associative binary operations of partition
1	0
2	30
3	240
4	240
5	360
6	300
7	60
8	2340
9	1380
10	840
11	360
12	240
13	1800
14	2400
15	420
16	540
17	0
18	1260
19	660
20	240
21	360
22	12480
23	7680
24	2160
25	43300
26	18600
27	1560
28	5220
29	4620
30	3120
31	2460
32	900
33	540
34	3240
35	1060
36	1440
37	4300
38	3460
39	260
40	2380
41	21545
42	11490
43	9465
44	4630
45	3215
46	536
47	1
Total	183732

7. CONCLUSIONS

The conclusion of this paper is that among the **298,023,223,876,953,125** different operations on a Five-element set, S={0,1,2,3,4}.

8. FUTURE WORK

One can find out Associative binary Operations on a n-Element Set by using same Algorithm , which we have used for Five-element and also verify the one of the result for six element set is **17061118** [7].

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