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Comparison of Adomian Decomposition and Taylor Expansion Methods for the Solutions of Fractional Integro-Differential Equations

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ABSTRACT
In this paper will be compared between Adomian decomposition method (ADM) and Taylor expansion method (TEM) for solving (approximately) a class of fractional integro-differential equations. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

General Terms
Numerical solutions, Fractional integro-differential equations.

Keywords
Fractional integro-differential equations, Adomian decomposition method, Taylor expansion method, Caputo fractional derivative, Riemann-Liouville.

1. INTRODUCTION
In this paper will be taken the fractional integro-differential equations with a Caputo fractional derivative of the type

$$D_*^q y(t) = P(t)y(t) + f(t) + \int_0^t K(t,x)y(x)dx, t \in [0,1] \quad (1.1)$$

with initial condition

$$y(0) = \alpha, \qquad n-1 < q \le n, n \in N \quad (1.2)$$

where D_*^q is Caputo's fractional derivative and α is a parameter describing the order of the fractional derivative. Such kind of equations arise in the mathematical modelling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems [6,11,16]. So far, fractional calculus as well as fractional differential equations have received increasing attention in recent years. The existence and uniqueness of solutions to fractional differential equations have been investigated [2,7,9,14]. In addition, when $\alpha \in N$, Eq(1.1) reduces to linear integro-differential equation and the numerical methods for this equation have been extensively studied by many authors [8,12,15]. There are many methods for seeking approximate solutions such as collocation method, the fractional differential transform method, Legendre wavelet method, Taylor expansion method and Adomian decomposition method, See ([1,4,5,10,13]). The outline of this paper is as follows: In section 2, we present some definitions. Section 3, contains the application of the Adomian decomposition method. Section 4, contains the application of the Taylor expansion method. Finally, Sec.5 devoted to illustrate some numerical examples on mentioned methods.

2. SOME DEFINITIONS AND NOTATIONS
Definition 2.1. A real function $f(x), x > 0$, is said to be in the space $C_\alpha, \alpha \in R$, if there exists a real number $p > \alpha$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$.

Definition 2.2. A real function $f(x), x > 0$, is said to be in the space $C_\alpha^k, k \in N \cup \{0\}$, if $f^k \in C_\alpha$.

Definition 2.3. [7] D^q (q is real) denotes the fractional differential operator of order q in the sense of Riemann-Liouville, defined by

$$D^q f(t) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t \dfrac{f(x)}{(t-x)^{q-n+1}} dx, 0 \le n-1 < q \le n, \\ \\ \dfrac{d^n f(t)}{dt^n}, q = n \in N. \end{cases} \quad (2.1)$$

Definition 2.4. [7] I^q denotes the fractional integral operator of order q in the sense of Riemann-Liouville, defined by

$$I^q f(t) = D^{-q} f(t) = \begin{cases} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t \dfrac{f(x)}{(t-x)^{1-q}} dx, q > 0, \\ \\ f(t), q = 0. \end{cases} \quad (2.2)$$

Definition 2.5. [7] Let $f \in C_{-1}^n, n \in N$. Then the Caputo fractional derivative of $f(t)$, defined by

$$D_*^q f(t) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \displaystyle\int_0^t \dfrac{f^{(n)}(x)}{(t-x)^{q-n+1}} dx, 0 \le n-1 < q \le n, \\ \\ \dfrac{d^n f(t)}{dt^n}, q = n \in N. \end{cases} \quad (2.3)$$

Now will be introduced some basic properties of fractional operator are listed below [7]:for $f \in C_\alpha, \alpha \geq -1, \mu \geq 1, \eta \geq 0, \beta > -1, \delta \geq 0$:

(I)$I^\mu \in C_0$.

(II)$I^\eta I^\delta f(x) = I^{\eta+\delta} f(x) = I^\delta I^\eta f(x)$.

(III)$D^\eta D^\delta f(x) = D^{\eta+\delta} f(x)$.

(IV)$D^\delta I^\delta f(x) = f(x)$.

(V)$I^\delta D_*^\delta f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, 0 \leq n-1 < \delta \leq n \in N$.

(VI)$I^\delta x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\delta)} x^{\beta+\delta}; x > 0$.

(VII)$D^\delta x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\delta)} x^{\beta-\delta}; x > 0$.

3. ADOMIAN DECOMPOSITION METHOD

Consider the equation (1.1) with initial condition (1.2) where D_* is the operator defined in (2.3).Operating with I^q on both sides of the equation (1.1) as follows:

$$y(t) = \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{t^k}{k!} + I^q \left[P(t)y(t) + f(t) + \int_0^t K(t,x)y(x)dx\right]. \quad (3.1)$$

Adomain decomposition method defines the solution by the series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \quad (3.2)$$

where the components $y_0, y_1, y_2, \ldots\ldots$ are determined recursively by

$$y_0(t) = \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{t^k}{k!} + I^q f(t), \quad (3.3)$$

$$y_{k+1}(t) = I^\alpha[P(t)y_k(t)] + I^q\left[\int_0^t K(t,x)y_k(x)dx\right]. \quad (3.4)$$

Decomposition method suggests that 0^{th} component $y_0(t)$ be defined by the initial conditions and the function $f(t)$ as described above. The other components namely $y_1, y_2, \ldots$etc. are derived recurrently.

4. TAYLOR EXPANSION METHOD

Consider the equation (1.1) with initial condition (1.2). First, integrate both sides of the equation (1.1) with respect to t for n times. Using Def. (2.4) as follows:

$$\int_0^t \frac{(t-x)^{n-q-1}}{\Gamma(n-q)} y(x)dx = \int_0^t \frac{(t-x)^{n-1}P(x)}{(n-1)!} y(x)dx + \int_0^t \frac{(t-x)^{n-1}f(x)}{(n-1)!}dx + \frac{1}{(n-1)!}\int_0^t y(x)\int_x^t K(s,x)(s-x)^{n-1}dsdx + Q_n(t). \quad (4.1)$$

Next, Assume that $y \in C^{m+1}[0,1], y(x)$ can be represented as Taylor expansion m^{th} order as follows:

$$y(x) = y(t) + y'(t) + \ldots\ldots + y^{(m)}(t)\frac{(x-t)^m}{m!} + \frac{y^{(m+1)}(\eta(x))}{(m+1)!}(x-t)^{m+1}. \quad (4.2)$$

Where $t < \eta(x) < x$. It is readily shown that the Lagrange reminder $\frac{y^{(m+1)}(\eta(x))}{(m+1)!}(x-t)^{m+1}$ is sufficiently small for large enough m provided that $y^{(m+1)}(x)$ is uniformly bounded function for any m on the interval [0,1]. So, we will neglect the remainder and the truncated Taylor expansion y (x) as follows:

$$y(x) \approx \sum_{j=0}^{m} y^{(j)}(t)\frac{(x-t)^j}{j!}. \quad (4.3)$$

Substituting the approximate expression (4.3) for y(x) into equation (4.1),

$$\sum_{j=0}^{m}\int_0^t \frac{(t-x)^{n-q-1}}{\Gamma(n-q)} y^{(j)}(t)\frac{(x-t)^j}{j!}dx = \sum_{j=0}^{m}\int_0^t \frac{(t-x)^{n-1}P(x)}{(n-1)!} y^{(j)}(t)\frac{(x-t)^j}{j!}dx + \int_0^t \frac{(t-x)^{n-1}f(x)}{(n-1)!}dx + \sum_{j=0}^{m}\frac{1}{(n-1)!}\int_0^t y^{(j)}(t)\frac{(x-t)^j}{j!}\int_x^t K(s,x)(s-x)^{n-1}dsdx + Q_n(t). \quad (4.4)$$

Or further;

$$k_{00}(t)y(t) + k_{01}(t)y'(t) + \ldots\ldots\ldots + k_{0m}(t)y^{(m)}(t) = f_{(n)}(t), \quad (4.5)$$

where

$$k_{0j} = \frac{(-1)^j t^{n+j-q}}{(n+j-q)\Gamma(n-q)j!} - \frac{1}{(n-1)!j!}\int_0^t (x-t)^j \int_x^t K(s,x)(s-x)^{n-1}dsdx - \frac{(-1)^j}{(n-1)!}\int_0^t P(x)(t-x)^{n+j-1}dx,\ j = 0,1,\ldots,m, \quad (4.6)$$

$$f_{(n)}(t) = \frac{1}{(n-1)!}\int_0^t (t-x)^{n-1}f(x) + Q_n(t). \quad (4.7)$$

Thus equation (4.1) becomes an m^{th} order, linear, ordinary differential equation with variable coefficients for y(t) and it 's derivative up to m. So, will be determined $y(t), \ldots, y^{(m)}(t)$by solving linear equations. Now, other m independent linear equations for $y(t), \ldots, y^{(m)}(t)$ are needed. This can be achieved by integrating both sides of the equation (4.1) with respect to t from 0 to s and changing the order of the integration.

$$\int_0^t \frac{(t-x)^{n-q}}{\Gamma(n+1-q)} y(x)dx = \int_0^t \frac{(t-x)^n P(x)}{(n)!} y(x)dx + \int_0^t \left(\frac{(t-x)^n f(x)}{(n)!} + Q_n(x)\right)dx + \frac{1}{(n)!}\int_0^t y(x)\int_x^t K(s,x)(s-x)^n dsdx. \quad (4.8)$$

Where replace variable s with t. Applying Taylor expansion again and substituting (4.3) for y(x) into equation (4.8) gives

$$k_{10}(t)y(t) + k_{11}(t)y'(t) + \dots\dots\dots + k_{1m}(t)y^{(m)}(t) = f_{(n+1)}(t), \quad (4.9)$$

where

$$k_{1j} = \frac{(-1)^j t^{n+j+1-q}}{(n+j+1-q)\Gamma(n+1-q)j!} - \frac{1}{n!j!}\int_0^t (x-t)^j \int_x^t K(s,x)(s-x)^n ds dx - \frac{(-1)^j}{n!j!}\int_0^t P(x)(t-x)^{n+j} dx, j = 0,1,\dots,m, \quad (4.10)$$

$$f_{(n+1)}(t) = \int_0^t \left(\frac{(t-x)^n f(x)}{n!} + Q_n(x)\right) dx. \quad (4.11)$$

By repeating the above integration process for i $(i \in N^+, 1 < i \le m)$ times.

$$k_{i0}(t)y(t) + k_{i1}(t)y'(t) + \dots\dots\dots + k_{im}(t)y^{(m)}(t) = f_{(n+i)}(t), \quad (4.12)$$

where

$$k_{ij} = \frac{(-1)^j t^{n+j+i-q}}{(n+j+i-q)\Gamma(n+i-q)j!} - \frac{1}{(n+i-1)!j!}\int_0^t (x-t)^j \int_x^t K(s,x)(s-x)^{n+i-1} ds dx - \frac{(-1)^j}{(n+i-1)!j!}\int_0^t P(x)(t-x)^{n+j+i-1} dx, \quad (4.13)$$

$$f_{(r)}(t) = \int_0^t f_{(r-1)}(x) dx, r > n+1, r \in N^+. \quad (4.14)$$

Therefore, equations (4.6), (4.9), (4.12) form a system (m+1) linear equations for (m+1) unknown functions $y(t), \dots, y^{(m)}(t)$. For simplicity, the system can be written as:

$$K_{mm}(t)Y_m(t) = F_m(t). \quad (4.15)$$

Where $K_{mm}(t)$ is an $(m+1)\times(m+1)$ square matrix function in t, $Y_m(t), F_m(t)$ are two vectors of length (m+1), and these are defined as

$$K_{mm}(t) = \begin{pmatrix} K_{00}(t) & K_{01}(t) & \dots & K_{0m}(t) \\ K_{10}(t) & K_{11}(t) & \dots & K_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{m0}(t) & K_{m1}(t) & \dots & K_{mm}(t) \end{pmatrix}, \quad (4.16)$$

$$Y_m(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m)}(t) \end{pmatrix}, F_m(t) = \begin{pmatrix} f_{(n)}(t) \\ f_{(n+1)}(t) \\ \vdots \\ f_{(n+m)}(t) \end{pmatrix}. \quad (4.17)$$

By using Cramer's rule, obtain the approximate solution y(t) as:

$$y(t) = \frac{\det(M_{mm}(t))}{\det(K_{mm}(t))}. \quad (4.18)$$

Where

$$M_{mm}(t) = \begin{pmatrix} f_n(t) & K_{01}(t) & \dots & K_{0m}(t) \\ f_{n+1}(t) & K_{11}(t) & \dots & K_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+m}(t) & K_{m1}(t) & \dots & K_{mm}(t) \end{pmatrix}. \quad (4.19)$$

5. NUMERICAL EXAMPLES

In this paper, according to Adomian decomposition method, choose m=3.So, calculate the numerical results $y_{App.(t)}$ by using equations (3.3), (3.4) and the absolute errors can be written as $e = |y_{App.}(t) - y_{exact}(t)|$, where $y_{App.}(t) = y_0(t) + y_1(t) + y_2(t)$. While, according to Taylor's method, choose m=3. So, calculate the numerical results $y_{App.}(t)$ by using equations (4.18) and the absolute errors can be written as $e = |y_{App.}(t) - y_{exact}(t)|$. All results are obtained by using Maple 16.

Example 5.1. Consider the fractional integro-differential equation :

$$D^{0.5}y(t) = \frac{t^{0.5}}{\Gamma(1.5)} + \frac{1}{2}te^t\left(\sin(t) - \cos(t) + \frac{\sin(t)}{t}\right) - e^t\sin(t)\,y(t) + \int_0^t e^s\cos(s)y(s)ds,$$

$$y(0) = 0. \quad (5.1)$$

With the exact solution $y(t) = t$. (Show table 1, and figure 1.)

Table 1. The absolute errors of Example 5.1.

t	Abs.E (ADM)	Abs.E (TEM)
0.1	2.108E-7	1.0E-10
0.2	1.15952E-5	1.12E-8
0.3	1.322509E-4	3.0E-9
0.4	7.931674E-4	2.7E-9
0.5	3.3390101E-3	2.0E-9
0.6	1.12024541E-2	2.67E-8
0.7	3.20227646E-2	3.35E-8
0.8	8.11245207E-2	3.38E-8
0.9	1.86775703E-1	6.43E-8
1.0	3.97547118E-1	0

To avoid difficult fractional integral, take the truncated Taylor expansions for $\cos(t), \sin(t), e^t$ in the equation (5.1).But, by using modified decomposition method [3], and select $y_0(t) = t$, obtain

$$y_1(t) = I^{0.5}(\frac{1}{2}te^t\sin(t) - \frac{1}{2}te^t\cos(t) - \frac{1}{2}e^t\sin(t)) + I^{0.5}(e^t\sin(t))y_0(t)) + I^{0.5}(\int_0^t e^s\cos y_0(s)ds) = 0,$$

$y_{n+2}(t) = 0, n \geq 0.$

So that the solution in closed form $y(t) = t$.

Example 5.2. Consider the fractional integro-differential equation :

$$D^{0.75}y(t) = \frac{t^{0.25}}{\Gamma(1.25)} - t^2 - \frac{t^4}{3} + ty(t) + \int_0^t tsy(s)ds,$$

$$y(0) = 0. \qquad (5.2)$$

With the exact solution $y(t) = t$. (Show table 2, and figure 2.)

Table 2. The absolute errors of Example 5.2.

t	Abs.E (ADM)	Abs.E (TEM)
0.1	2.019E-8	1.0354E-7
0.2	1.5524E-6	2.7244E-6
0.3	1.99559E-5	1.81447E-5
0.4	1.237975E-4	6.81375E-5
0.5	5.168022E-4	1.844504E-4
0.6	1.6831889E-3	3.991721E-4
0.7	4.6270796E-3	7.196495E-4
0.8	1.12501689E-2	1.0759817E-3
0.9	2.49296030E-2	1.2179160E-3
1.0	5.13811204E-2	5.203903E-4

But, by using modified decomposition method [3], and select $y_0(t) = t$, obtain

$$y_1(t) = I^{0.75}(-t^2 - \frac{t^4}{3}) + I^{0.75}(ty_0(t)) + I^{0.75}(\int_0^t tsy_0(s)ds) = 0,$$

$y_2(t) = 0,$

$y_{n+3}(t) = 0, n \geq 0.$

So that the solution in closed form $y(t) = t$.

Example 5.3. Consider the fractional integro-differential equation :

$$D^{0.5}y(t) = (\cos(t) - \sin(t))y(t) + f(t) + \int_0^t t\sin(s)y(s)ds,$$

$$y(0) = 0. \qquad (5.3)$$

$$f(t) = \frac{2t^{1.5}}{\Gamma(2.5)} + \frac{t^{0.5}}{\Gamma(1.5)} + t(2 - 3\cos(t) - t\sin(t) + t^2\cos(t))$$

With the exact solution $y(t) = t^2 + t$. (Show table 3, and figure 3.)

Table3. The absolute errors of Example 5.3.

t	Abs.E (ADM)	Abs.E (TEM)
0.1	7.982452E-4	5.626E-7
0.2	3.6471117E-3	1.29901E-5
0.3	7.8054711E-3	8.24971E-5
0.4	1.19599592E-2	3.084884E-4
0.5	1.50115353E-2	8.647433E-4
0.6	1.64762039E-2	2.0151158E-3
0.7	1.6551653E-2	4.123505E-3
0.8	1.5885982E-2	7.650717E-3
0.9	1.5148298E-2	1.3138091E-2
1.0	1.4554082E-2	2.1177649E-2

To avoid difficult fractional integral take the truncated Taylor expansions for the trigonometric terms in terms in the equation (5.3) e.g. $[\cos t \approx 1 - \frac{t^2}{2!} + \frac{t^4}{4!}, \sin t \approx t - \frac{t^3}{3!} + \frac{t^5}{5!}]$. But, by using a modified decomposition method [3], and select $y_0(t) = t^2 + t$, obtain

$$y_1(t) = I^{0.5}(2t - 3t\cos(t) - t^2\sin(t) + t^3\cos(t)) + I^{0.5}((\cos(t) - \sin(t))y_0(t)) + I^{0.5}(\int_0^t t\sin(s)y_0(s)ds) = 0,$$

$y_{n+2}(t) = 0, n \geq 0.$

So that the solution in closed form $y(t) = t^2 + t$.

Example 5.4. Consider the fractional integro-differential equation :

$$D^{0.25}y(t) = \frac{6t^{2.75}}{\Gamma(3.75)} - \frac{1}{5}t^2e^ty(t) + \int_0^t e^tsy(s)ds,$$

$$y(0) = 0. \qquad (5.4)$$

With the exact solution $y(t) = t^3$. (Show table 4. and figure 4.)

Table 4. The absolute errors of Example 5.4.

t	Abs.E (ADM)	Abs.E (TEM)
0.1	**0**	7.2341E-9
0.2	**0**	5.97738E-7
0.3	**0**	7.97449E-6
0.4	**0**	5.173697E-5
0.5	**0**	2.224414E-4
0.6	**0**	7.350817E-4
0.7	**0**	2.0168252E-3
0.8	**0**	4.8069379E-3
0.9	**0**	1.02260626E-2
1.0	**0**	1.97594433E-2

According to Adomian decomposition method, find the exact solution because $y_0(t) = t^3$.

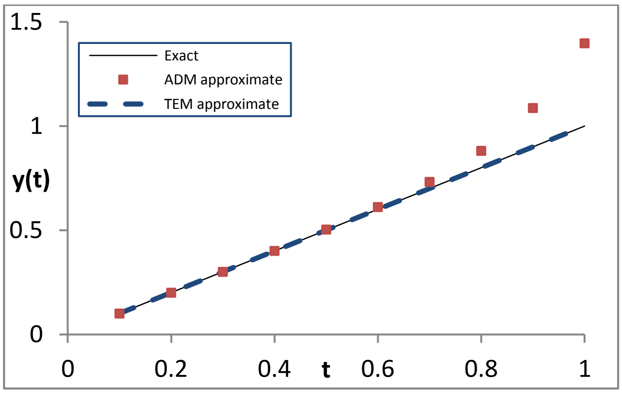

Fig.1. Comparison of approximate solutions obtained by ADM and TEM with exact solution of example 5.1.

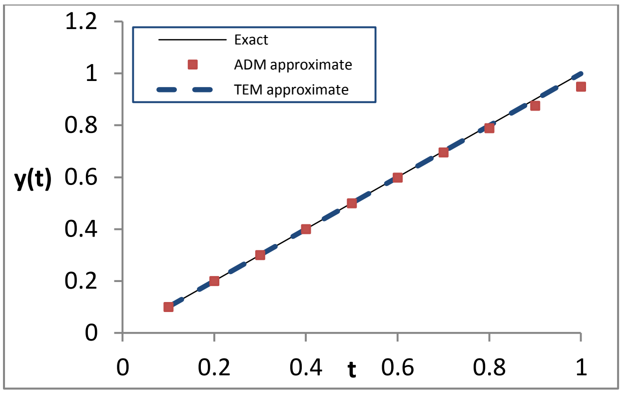

Fig.2. Comparison of approximate solutions obtained by ADM and TEM with exact solution of example 5.2.

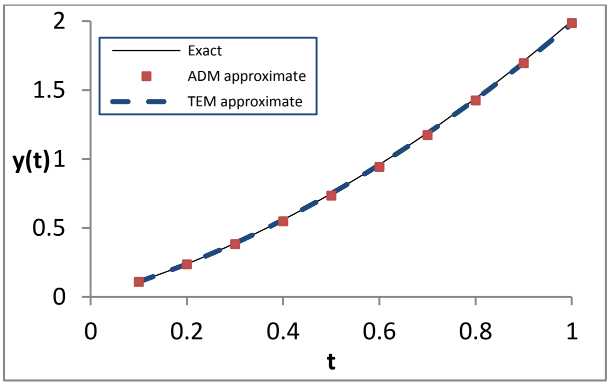

Fig.3. Comparison of approximate solutions obtained by ADM and TEM with exact solution of example 5.3.

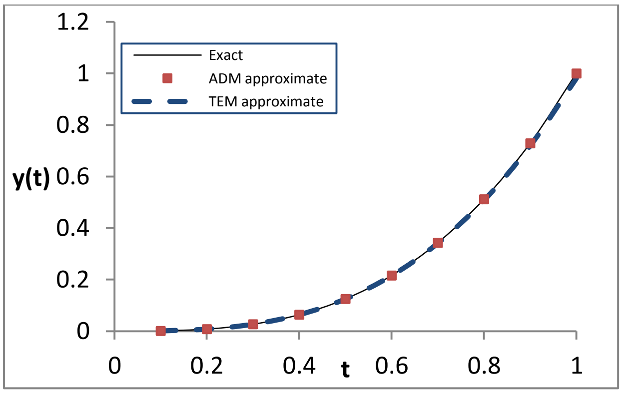

Fig.4. Comparison of approximate solutions obtained by ADM and TEM with exact solution of example 5.4.

6. CONCLUSION

In this paper, this study showed that for most problems the results in TEM are better than the results in ADM (see examples 1, 2, and 3). Except in the case of $y_0(t)$ equal to exact solution, find the results in ADM are the exact Solution (see example 4).

7. ACKNOWLEDGMENTS

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