Data Dependence of Some New Iterative Schemes for Quasi-Contractive Operators

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ABSTRACT

In this paper we prove data dependence of new multistep iterative scheme as well as CR iterative scheme for quasi contractive operators, that is, by using an approximate quasi - contractive operator we approximate the fixed point of the given operators.

KEYWORDS:
CR Iteration, Data Dependence, New Multistep Iteration, Quasi Contractive.

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1. INTRODUCTION

Fixed point theory is involved in various types of issues such as to find the fixed point, existence and uniqueness of fixed point etc. Data dependence of fixed point is one of these issues and has become a subject of research interest from some time now. The data dependence of various iterative schemes has been studied by various authors like Rus and Muresan [8], Rus et al. [6, 7], Berinde[24], Espinola and Petrusel [17], Markin [10], Chifu and Petrusel [4], Olantiwo [11,12], Soltuz [19, 20], Soltuz and Grosan [21], Chugh and Kumar [15], Gursoy et al.[5] and several references thereof.

Our main interest in this paper is to show data dependence of new multi step [2] and CR [16] iterative schemes using quasi contractive operators. For the background of our exposition, we first mention some contractive mappings.

Zemfirescu [22] established a nice generalization of the Banach fixed point theorem as follows:

Let (X, d) be a complete metric space and T: X→X mappings for which there exists real numbers α,β,γ satisfying $0 \le \alpha < 1, 0 \le \beta < \frac{1}{2}, 0 \le \gamma < \frac{1}{2}$, respectively such that for each $x, y \in E$, at least one of the following is true:

$$\begin{aligned} &(z_1)\ d(Tx,Ty) \le \alpha d(x,y) \\ &(z_2)\ d(Tx,Ty) \le \beta[d(x,Tx)+d(y,Ty)] \qquad (1.1) \\ &(z_3)\ d(Tx,Ty) \le \gamma[d(x,Ty)+d(y,Tx)] \end{aligned}$$

Then the mapping T satisfying (1.1) is called Zamfirescu contraction.

Remark 1.1. Mapping which satisfy (z_2) is called a Kannan mapping, while the mapping satisfying (z_3) is called Chatterjea operator.

The contractive condition (1.1) implies

$$\begin{cases} \|Tx-Ty\| \le 2\delta\|x-Tx\|+\delta\|x-y\| \\ \text{and} \\ \|Tx-Ty\| \le 2\delta\|x-Ty\|+\delta\|x-y\| \end{cases} \text{, for all } \forall\, x,y \in E ,$$

where $\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, $0 \le \delta < 1$. (1.2)

Osilike and Udomene [14] defined a new general definition of quasi contractive operator as follows:

$\|Tx-Ty\| \le \delta\|x-y\| + L\|x-Tx\| \quad \forall\, x, y \in E$ and some $L \ge 0,\ \delta \in [0,1]$. (1.3)

After that a more general definition was introduced by Imoru and Olantiwo [3] as follows: if there exists a constant $0 \le \delta < 1$ and a monotonically increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, such that for $\forall\, x, y \in E$

$$\|Tx-Ty\| \le \delta\|x-y\| + \varphi(\|x-Tx\|) \qquad (1.4)$$

Now we recapitulate some iterative schemes in the literature of fixed point theory. Let X be a Banach space and E be a closed, convex subset of X. If T: X→X a mappings on E, then $F_T = \{p \in X : Tp = p\}$ denotes the set of fixed points of T.

For $x_0 \in E$, Ishikawa Iteration [18], $\{x_n\}_{n=0}^{\infty}$ is defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n , \end{aligned} \qquad (1.5)$$

(1.5) where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Observe that if $\beta_n = 0$ for each *n*, then the Ishikawa iteration process (1.5) reduces to the Mann iteration scheme.

For $x_0 \in E$, the Noor three step iterative scheme [13], $\{x_n\}_{n=0}^{\infty}$ is defined as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + T y_n \\ y_n &= (1-\beta_n)x_n + T z_n \\ z_n &= (1-\gamma_n)x_n + T x_n , \end{aligned} \qquad (1.6)$$

(1.6) where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0, 1] satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$.

If γ_n=0, then Noor iteration process (1.6) reduces to Ishikawa Iteration scheme (1.5).

In 2007, Agarwal et al. defined the Agarwal et al. iterative scheme [1] as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n , \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0, 1] with $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Quite recently, Phuengrattana and Suantai [25] introduced SP iterative scheme as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n , \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0, 1] satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$.

We shall use the following iterative schemes:

(a) **New multi step iterative scheme**

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n^1 \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T_1 y_n^{i+1}, i = 1,2....k-2 \\ y_n^{p-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T_1 x_n \end{cases}$$

(1.9)

and

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n T_2 v_n^1 \\ v_n^i = (1-\beta_n^i)u_n + \beta_n^i T_2 v_n^{i+1}, i = 1,2....k-2 \\ v_n^{p-1} = (1-\beta_n^{k-1})u_n + \beta_n^{k-1} T_2 u_n, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}, i = 1,..k-2, k \geq 2$ be the real sequences of positive numbers in [0, 1] satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$, due to Rhoades and Soltuz[2].

(b) **CR iterative scheme**

$$\begin{cases} x_0 \in E \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)T_1 x_n + \beta_n T_1 z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_1 x_n \end{cases},$$

and

(1.10)

$$\begin{cases} u_{n+1} = (1-\alpha_n)v_n + \alpha_n T_2 v_n \\ v_n = (1-\beta_n)T_2 u_n + \beta_n T_2 w_n \\ w_n = (1-\gamma_n)u_n + \gamma_n T_2 u_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0, 1] satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$, due to Chugh and Kumar [16].

Remark 1.2. By putting k=2 and 3 , (1.9) reduces to (1.5) and (1.6) respectively.

Now to prove our main results we need following results in sequel.

Definition 1.3. [23] Let T_1, T_2 be two operators. We say T_2 is approximate operator of T_1 if for all $x \in X$ and for a fixed $\varepsilon > 0$, we have $\|T_1 x - T_2 x\| \leq \varepsilon$.

Lemma 1.4.[21] Let $\{\alpha_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one suppose there exists $n_0 \in I$, such that for all $n \geq n_0$ it satisfies the following inequality:

$\alpha_n \leq (1-\lambda_n)\alpha_n + \lambda_n \sigma_n$,

where $\lambda_n \in (0,1)$, $\forall n \in N, \sum_{n=1}^{\infty}\lambda_n = \infty$ and $\sigma_n \geq 0 \forall n \in N$.

Then, $0 \leq \limsup_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\sigma_n$.

Theorem 1.5. [16]: Let T: E→E be a mapping satisfying (1.4) and $\{x_n\}_{n=0}^{\infty}$ be defined by (1.10) with real sequence, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$ .Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to unique fixed point of T.

Theorem 1.6.[9]: Let T: E→E be a mapping satisfying (1.4) and $\{x_n\}_{n=0}^{\infty}$ be defined by (1.9) with real sequence, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}, i = 1,..k-1$ in $[0,1)$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$ .Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to unique fixed point of T.

2. MAIN RESULTS

Theorem 2.1. Let $T_1 : E \to E$ be a mapping satisfying (1.4). Let T_2 be a approximate operator of T_1 as in Definition1and $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two CR iterative schemes defined by (1.10) associated to T_1 and T_2 , respectively, where, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1)$ satisfying $\begin{cases} (i)\dfrac{1}{2} \leq \alpha_n(1-\delta), \forall n \\ (ii)\sum_{n=0}^{\infty}\alpha_n = \infty \end{cases}$. Let $p = T_1 p$ and $q = T_2 q$ then we have the following estimate:

$$\|p - q\| \leq \frac{4\varepsilon}{1-\delta} .$$

Proof: Using (1.4) and (1.10), we have the following estimates:

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|(1-\alpha_n)(y_n-v_n)+\alpha_n(T_1y_n-T_2v_n)\|\\ &\le (1-\alpha_n)\|y_n-v_n\|+\alpha_n\|T_1y_n-T_2v_n\|\\ &= (1-\alpha_n)\|y_n-v_n\|+\alpha_n\|T_1y_n-T_1v_n+T_1v_n-T_2v_n\|\\ &\le (1-\alpha_n)\|y_n-v_n\|+\alpha_n\left\{\|T_1y_n-T_1v_n\|+\|T_1v_n-T_2v_n\|\right\}\\ &\le (1-\alpha_n)\|y_n-v_n\|+\alpha_n\left\{\delta\|y_n-v_n\|+\varphi\left(\|T_1y_n-y_n\|\right)+\varepsilon\right\}\\ &\le (1-\alpha_n(1-\delta))\|y_n-v_n\|+\alpha_n\varphi\left(\|T_1y_n-y_n\|\right)+\alpha_n\varepsilon, \end{aligned} \tag{2.1}$$

$$\begin{aligned}\|y_n-v_n\| &= \|(1-\beta_n)(T_1x_n-T_2u_n)+\beta_n(T_1z_n-T_2w_n)\|\\ &\le (1-\beta_n)\|T_1x_n-T_2u_n\|+\beta_n\|T_1z_n-T_2w_n\|\\ &\le (1-\beta_n)\|T_1x_n-T_1u_n+T_1u_n-T_2u_n\|\\ &\quad +\beta_n\|T_1z_n-T_1w_n+T_1w_n-T_2w_n\|\\ &\le (1-\beta_n)\left\{\|T_1x_n-T_1u_n\|+\|T_1u_n-T_2u_n\|\right\}\\ &\quad +\beta_n\left\{\|T_1z_n-T_1w_n\|+\|T_1w_n-T_2w_n\|\right\}\\ &\le (1-\beta_n)\left\{\delta\|x_n-u_n\|+\varphi\left(\|T_1x_n-x_n\|\right)+\varepsilon\right\}\\ &\quad +\beta_n\left\{\delta\|z_n-w_n\|+\varphi\left(\|T_1z_n-z_n\|\right)+\varepsilon\right\}\\ &= (1-\beta_n)\delta\|x_n-u_n\|+(1-\beta_n)\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +(1-\beta_n)\varepsilon+\beta_n\delta\|z_n-w_n\|+\beta_n\varphi\left(\|T_1z_n-z_n\|\right)+\beta_n\varepsilon \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\|z_n-w_n\| &= \|(1-\gamma_n)(x_n-u_n)+\gamma_n(T_1x_n-T_2u_n)\|\\ &\le (1-\gamma_n)\|x_n-u_n\|+\gamma_n\|T_1x_n-T_2u_n\|\\ &\le (1-\gamma_n)\|x_n-u_n\|+\gamma_n\left\{\delta\|x_n-u_n\|+\varphi\left(\|T_1x_n-x_n\|\right)+\varepsilon\right\}\\ &= (1-\gamma_n(1-\delta))\|x_n-u_n\|+\gamma_n\varphi\left(\|T_1x_n-x_n\|\right)+\gamma_n\varepsilon \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-\alpha_n(1-\delta))\{(1-\beta_n)\delta\|x_n-u_n\|\\ &\quad +(1-\beta_n)\varphi\left(\|T_1x_n-x_n\|\right)+(1-\beta_n)\varepsilon\}\\ &\quad +(1-\alpha_n(1-\delta))\left\{\beta_n\delta\|z_n-w_n\|+\beta_n\varphi\left(\|T_1z_n-z_n\|\right)+\beta_n\varepsilon\right\}\\ &\quad +\alpha_n\varphi\left(\|T_1y_n-y_n\|\right)+\alpha_n\varepsilon\\ &= (1-\alpha_n(1-\delta))(1-\beta_n)\delta\|x_n-u_n\|\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\delta\|z_n-w_n\|\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varepsilon\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varphi\left(\|T_1z_n-z_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varepsilon\\ &\quad +\alpha_n\varphi\left(\|T_1y_n-y_n\|\right)+\alpha_n\varepsilon\\ &= (1-\alpha_n(1-\delta))(1-\beta_n)\delta\|x_n-u_n\|\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\delta\{(1-\gamma_n(1-\delta))\|x_n-u_n\|\\ &\quad +\gamma_n\varphi\left(\|T_1x_n-x_n\|\right)+\gamma_n\varepsilon\}\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varepsilon\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varphi\left(\|T_1z_n-z_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varepsilon+\alpha_n\varphi\left(\|T_1y_n-y_n\|\right)+\alpha_n\varepsilon\\ &\le \left\{(1-\alpha_n(1-\delta))(1-\beta_n(1-\delta)(1-\gamma_n(1-\delta)))\right\}\|x_n-u_n\|\\ &\quad +\delta(1-\alpha_n(1-\delta))\beta_n\gamma_n\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\delta\gamma_n\varepsilon\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))(1-\beta_n)\varepsilon\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varphi\left(\|T_1z_n-z_n\|\right)\\ &\quad +(1-\alpha_n(1-\delta))\beta_n\varepsilon+\alpha_n\varphi\left(\|T_1y_n-y_n\|\right)\\ &\quad +\alpha_n\varepsilon. \end{aligned} \tag{2.4}$$

It may be noted that for $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \subset [0,1)$ and $0\le \delta <1$, the following inequalities hold:

$$\begin{cases}(1-\alpha_n)\delta<(1-\alpha_n),\\ (1-\beta_n(1-\delta)(1-\gamma_n(1-\delta)))<1,\\ \alpha_n\beta_n\delta<\alpha_n.\end{cases} \tag{2.5}$$

It follows from assumption (i) that

$$(1-\alpha_n(1-\delta))<\alpha_n(1-\delta)\le\alpha_n, \forall n\in I \tag{2.6}$$

Now using (2.5) and (2.6) in (2.4), we get

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-\alpha_n(1-\delta))\|x_n-u_n\|+2\alpha_n\varphi\left(\|T_1x_n-x_n\|\right)\\ &\quad +4\alpha_n\varepsilon+2\alpha_n\varphi\left(\|T_1z_n-z_n\|\right)+\alpha_n\varphi\left(\|T_1y_n-y_n\|\right), \end{aligned}$$

which further implies

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-\alpha_n(1-\delta))\|x_n-u_n\|\\ &\quad +\alpha_n(1-\delta)\frac{\{2\varphi\left(\|T_1x_n-x_n\|\right)+4\varepsilon+\varphi\left(\|T_1z_n-z_n\|\right)+\varphi\left(\|T_1y_n-y_n\|\right)\}}{(1-\delta)} \end{aligned} \tag{2.7}$$

Let us denote

$a_n = \|x_n - u_n\|$

$r_n = \alpha_n(1-\delta)$

and

$$\sigma_n = \frac{\{2\varphi(\|T_1x_n - x_n\|) + 4\varepsilon + \varphi(\|T_1z_n - z_n\|) + \varphi(\|T_1y_n - y_n\|)\}}{(1-\delta)}.$$

Now from Theorem 1 we have $\lim_{n\to\infty}\|x_n - p\| = 0$. Also T satisfies condition (1.4) and $Tp = p \in F_T$, hence

$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|y_n - Ty_n\| = \lim_{n\to\infty}\|z_n - Tz_n\| = 0$.

Because $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$ converges to fixed point of T

$\lim_{n\to\infty}\varphi(\|x_n - Tx_n\|) = \lim_{n\to\infty}\varphi(\|y_n - Ty_n\|) = \lim_{n\to\infty}\varphi(\|z_n - Tz_n\|) = 0$.

Since φ is continuous, hence using Lemma 1, (2.7) yields

$$\|p - q\| \le \frac{4\varepsilon}{(1-\delta)}.$$

Theorem 2.2. Let $T_1 : E \to E$ be a mapping satisfying (1.4). Let T_2 be a approximate operator of T_1 as in Definition 1 and $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two iterative schemes defined by (1.9) associated to T_1 and T_2 respectively, where, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}, i = 1,...k-1$ are real sequences in $[0,1)$ satisfying

$$\begin{cases} (i)\ o \le \beta_n^i \le \alpha_n < 1, i = 1,..k-1 \\ (ii)\sum_{n=0}^{\infty}\alpha_n = \infty \end{cases}$$

. Let $p = T_1p$ and $q = T_2q$ then we have the following estimate

$$\|p - q\| \le \frac{(k-1)\varepsilon}{1-\delta}.$$

Proof: For a given $x_0 \in E$ and $u_0 \in E$ we consider the following iterative schemes for T_1 and T_2

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n^1 \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T_1 y_n^{i+1}, i = 1,2....k-2 \\ y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1}T_1x_n, \end{cases} \quad \text{and} \quad (2.8)$$

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n T_2 v_n^1 \\ v_n^i = (1-\beta_n^i)u_n + \beta_n^i T_2 v_n^{i+1}, i = 1,2....k-2 \\ v_n^{p-1} = (1-\beta_n^{p-1})u_n + \beta_n^{p-1}T_2u_n, \end{cases} \quad (2.9)$$

then using (1.4), (2.8) and (2.9) , yield the following estimates:

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(T_1y_n^1 - T_2v_n^1)\| \\ &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|T_1y_n^1 - T_2v_n^1\| \\ &= (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|T_1y_n^1 - T_1v_n^1 + T_1v_n^1 - T_2v_n^1\| \\ &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|T_1y_n^1 - T_1v_n^1\| + \alpha_n\|T_1v_n^1 - T_2v_n^1\| \\ &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\delta\|y_n^1 - v_n^1\| + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) + \alpha_n\varepsilon \end{aligned}$$

, (2.10)

$$\begin{aligned} \|y_n^1 - v_n^1\| &= \|(1-\beta_n^1)(x_n - u_n) + \beta_n^1(T_1y_n^2 - T_2v_n^2)\| \\ &\le (1-\beta_n^1)\|x_n - u_n\| + \beta_n^1\|T_1y_n^2 - T_2v_n^2\| \\ &\le (1-\beta_n^1)\|x_n - u_n\| + \beta_n^1\delta\|T_1y_n^2 - T_2v_n^2\| + \beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) + \beta_n^1\varepsilon \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \|y_n^2 - v_n^2\| &= \|(1-\beta_n^2)(x_n - u_n) + \beta_n^2(T_1y_n^3 - T_2v_n^3)\| \\ &\le (1-\beta_n^2)\|x_n - u_n\| + \beta_n^2\|T_1y_n^3 - T_2v_n^3\| \\ &\le (1-\beta_n^2)\|x_n - u_n\| + \beta_n^2\delta\|y_n^3 - v_n^3\| + \beta_n^2\varphi(\|y_n^3 - T_1y_n^3\|) + \beta_n^2\varepsilon \end{aligned} \quad (2.12)$$

Combining (2.10), (2.11) and (2.12), we have

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) + \alpha_n\varepsilon \\ &\quad + \alpha_n\delta\{(1-\beta_n^1)\|x_n - u_n\| + \beta_n^1\delta\|y_n^2 - v_n^2\| \\ &\quad + \beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) + \beta_n^1\varepsilon\} \\ &\le \left[(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1\right]\|x_n - u_n\| \\ &\quad + \delta^2\alpha_n\beta_n^1\|y_n^2 - v_n^2\| + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) \\ &\quad + \delta\alpha_n\beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) + \alpha_n\varepsilon + \delta\alpha_n\beta_n^1\varepsilon \\ &\le \left[(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2))\right]\|x_n - u_n\| \\ &\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\|y_n^3 - v_n^3\| + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) \\ &\quad + \delta\alpha_n\beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) + \delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|y_n^3 - T_1y_n^3\|) \\ &\quad + \alpha_n\varepsilon + \delta\alpha_n\beta_n^1\varepsilon + \delta\alpha_n\beta_n^1\beta_n^2 \end{aligned} \quad (2.13)$$

Thus inductively, we get

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| \le\ & [(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2) - \delta^2\beta_n^2(1-\beta_n^3) - \\ & .... - \delta^{p-4}\beta_n^2\beta_n^3...\beta_n^{k-4}(1-\beta_n^{k-3}))]\|x_n - u_n\| \\ & + \delta^{k-3}\alpha_n\beta_n^1\beta_n^2...\beta_n^{k-4}\|y_n^{k-2} - v_n^{k-2}\| \\ & + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) + \delta\alpha_n\beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) \\ & + \delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|y_n^3 - T_1y_n^3\|) + \\ & ...... + \delta^{k-3}\alpha_n\beta_n^1\beta_n^2...\beta_n^{k-3}\varphi(\|y_n^{k-2} - T_1y_n^{k-2}\|) \\ & + \alpha_n\varepsilon + \delta\alpha_n\beta_n^1\varepsilon + \delta\alpha_n\beta_n^1\beta_n^2\varepsilon + \\ & .... + \delta^{k-3}\alpha_n\beta_n^1\beta_n^2...\beta_n^{k-3}\varepsilon \end{aligned} \quad (2.14)$$

Using (1.4) and (1.9)

$$\begin{aligned} \|y_n^{k-2} - v_n^{k-2}\| &\le \|(1-\beta_n^{k-2})(x_n - u_n) + \beta_n^{k-2}(T_1x_n - T_2u_n)\| \\ &\le (1-\beta_n^{k-2})\|x_n - u_n\| + \beta_n^{k-2}\|T_1x_n - T_2u_n\| \\ &\le (1-\beta_n^{k-2}(1-\delta))\|x_n - u_n\| + \beta_n^{k-2}\varphi(\|x_n - T_1x_n\|) + \beta_n^{k-2}\varepsilon \end{aligned} \quad (2.15)$$

Now by combining (2.14) and (2.15)

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| \le\ & [(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2) - \\ & .... - \delta^{k-4}\beta_n^2...\beta_n^{k-3}(1-\beta_n^{k-2}(1-\delta))]\|x_n - u_n\| \\ & + \alpha_n\varphi(\|y_n^1 - T_1y_n^1\|) + \delta\alpha_n\beta_n^1\varphi(\|y_n^2 - T_1y_n^2\|) \\ & + \delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|y_n^3 - T_1y_n^3\|) + \\ & ...... + \delta^{k-2}\alpha_n\beta_n^1...\beta_n^{k-2}\varphi(\|x_n - T_1x_n\|) \\ & + \alpha_n\varepsilon + \delta\alpha_n\beta_n^1\varepsilon + ... + \delta^{k-2}\alpha_n\beta_n^1...\beta_n^{k-2}\varepsilon, \end{aligned}$$

which further implies

$$\|x_{n+1}-u_{n+1}\| \le \left[(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1+L\right]\|x_n-u_n\| + \alpha_n\varphi(\|y_n^1-T_1y_n^1\|)+\delta\alpha_n\beta_n^1\varphi(\|y_n^2-T_1y_n^2\|) + \delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|y_n^3-T_1y_n^3\|)+\ldots\ldots + \delta^{k-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\varphi(\|x_n-T_1x_n\|) + \alpha_n\varepsilon+\delta\alpha_n\beta_n^1\varepsilon+\ldots.+\delta^{p-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\varepsilon, \quad (2.16)$$

where

$$\begin{cases} L=\delta^2\alpha_n\beta_n^1(1-\beta_n^2)\ldots.+\delta^{p-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{p-3}(1-\beta_n^{k-2}(1-\delta)) \\ \le \delta^2\alpha_n\beta_n^1\beta_n^2\ldots.+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-3} \\ \le \delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots.+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-3} \\ \le \delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\ldots.+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2 \\ =\delta^2\alpha_n\beta_n^1\beta_n^2[1+\delta+\delta^2+\ldots\delta^{k-5}] \\ =\delta^2\alpha_n\beta_n^1\beta_n^2\frac{[1-\delta^{k-4}]}{[1-\delta]}<\delta\alpha_n\beta_n^1 \end{cases}$$

(2.17)

Since $\delta\frac{[1-\delta^{k-3}]}{[1-\delta]}<1$, this imply

$$\{(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1+L<(1-\alpha_n(1-\delta)). \quad (2.18)$$

Hence using (2.17) and (2.18) in (2.16), we get

$$\|x_{n+1}-u_{n+1}\| \le [1-(1-\alpha_n(1-\delta))]\|x_n-u_n\| + \alpha_n\varphi(\|y_n^1-T_1y_n^1\|)+\alpha_n\varphi(\|y_n^2-T_1y_n^2\|) + \alpha_n\varphi(\|y_n^3-T_1y_n^3\|)+\ldots\ldots + \alpha_n\varphi(\|x_n-T_1x_n\|)+(k-1)\varepsilon$$

$$=[1-(1-\alpha_n(1-\delta))]\|x_n-u_n\| +\alpha_n(1-\delta)\left\{\frac{\varphi(\|y_n^1-T_1y_n^1\|)+\varphi(\|y_n^2-T_1y_n^2\|)+\ldots+\alpha_n\varphi(\|x_n-T_1x_n\|)+(k-1)\varepsilon}{(1-\delta)}\right\} \quad (2.19)$$

Let us denote

$$a_n=\|x_n-u_n\|$$

$$r_n=\alpha_n(1-\delta),$$

and

$$\sigma_n : \frac{\{\varphi(\|y_n^1-T_1y_n^1\|)+\varphi(\|y_n^2-T_1y_n^2\|)+\ldots\ldots+\varphi(\|x_n-T_1x_n\|)+(k-1)\varepsilon\}}{(1-\delta)}$$

.

Now from Theorem 2, we have $\lim_{n\to\infty}\|x_n-p\|=0$. Also T satisfies condition (1.4) and $Tp=p\in F_T$, using the similar argument as in Theorem 3, we get,

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|y_n^1-T_1y_n^1\|=\lim_{n\to\infty}\|y_n^2-T_1y_n^2\|=\ldots=\lim_{n\to\infty}\|y_n^{k-1}-T_1y_n^{k-1}\|=0$$

.

Since φ is continuous, we have

$$\lim_{n\to\infty}\varphi(\|x_n-T_1x_n\|)=\lim_{n\to\infty}\varphi(\|y_n^1-T_1y_n^1\|)=\lim_{n\to\infty}\varphi(\|y_n^2-T_1y_n^2\|) =..=\lim_{n\to\infty}\varphi(\|y_n^{k-1}-T_1y_n^{k-1}\|)=0$$

Hence using Lemma 1, (2.19) yields

$$\|p-q\|\le\frac{(k-1)\varepsilon}{1-\delta}.$$

Remark 2.3. Since the iteration (1.5) and (1.6) are special cases of iterative scheme (1.9), so. Theorem 2 generalizes existing result for (1.5) and (1.6). By taking k = 3 and k = 2 in Theorem 4, data dependence results for the iterative schemes (1.5) and (1.6) can be obtained easily.

Numerical example 2.1.[21]. Let T: R→R be a quasi – contractive operator defined by

$$T(x)=\begin{cases}0 & \text{if } x\in(-\infty,2] \\ -0.5 & \text{if } x\in(2,\infty)\end{cases},$$

with q=0.2, φ = identity and having unique fixed point zero.

Now consider mapping **S**: R→R defined by

$$S(x)=\begin{cases}1 & \text{if } x\in(-\infty,2] \\ -1.5 & \text{if } x\in(2,\infty)\end{cases},$$

with unique fixed point one. Let us assume ε =1, distance between two maps as follows:

$\|Sx-Tx\|\le 1, \forall x\in R$.

if $u_0=x_0=0$ and $\alpha_n,\beta_n,\gamma_n=\frac{1}{n+1}$ then by using executable program in C++ we get following table for CR and new multi step iterative schemes.

Table 2.1

Number of Iterative Step	New Multi step Iteration	Number of Iterative Step	CR Iteration
1	0.707107	1	1
2	0.914214	2	1
3	0.974874	3	1
4	0.992641	4	1
5	0.997845	-	1
6	0.999369	-	-
7	0.999815	-	-
8	0.999946	-	-
9	0.999984	-	-
10	0.999995	-	-
11	0.999999		
12	1		
13	1		

When new multi step and CR iterative schemes applied to S it converges to unique fixed point x^*=1.Also the distance between the

fixed point of S and T is one therefore if $\varepsilon = \frac{1}{4}$ then without computing the fixed point of S, from theorem 3.1 we have the following estimates:

$$\|x^* - u^*\| \le \frac{1}{1-q} = \frac{1}{1-0.2} = 1.2 .$$

Also, if $\varepsilon = \frac{1}{1-k}$, then from theorem 3.2 we have the following estimates:

$$\|x^* - u^*\| \le \frac{1}{1-q} = \frac{1}{1-0.2} = 1.2 .$$

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