Inference of a Common Fixed Point Theorem on Four Self Maps

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ABSTRACT:
The aim of this paper is to prove a common fixed point theorem which generalizes the result of Brian Fisher [1] and etal. by weaker conditions. The conditions of continuity, compatibility and completeness of a metric space are replaced by weaker conditions such as weakly compatible and the associated sequence.

KEYWORDS:
Fixed point, self maps, compatible maps, weakly compatible mappings, associated sequence.

I. INTRODUCTION
Two self maps S and T are said to be commutative if ST = TS. The concept of the commutativity has been generalized in several ways. For this Gerald Jungck [2] initiated the concept of compatibility.

1.1 Compatible Mappings:
Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It can be easily verified that when the two mappings are commuting then they are compatible but not conversely.

In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely.

1.2. Weakly Compatible:
A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

Brian Fisher and others [1] proved the following common Fixed Point theorem for four self maps of a complete metric space.

Theorem 1.3 Suppose A, B, S and T are four self maps of metric space(X,d)such that

1.3.1 (X,d) is a complete metric space

1.3.2 $A(x)\subseteq T(x)$, $B(x)\subseteq S(x)$

1.3.3 The pairs (A,S) and,(B,T) are compatible

1.3.4 $d(Ax,By)^2 < c_1 \max\{d(Sx,Ax)^2, d(Ty,By)^2, d(Sx,Ty)^2\}$
$+c_2 \max\{d(Sx,Ax)d(Sx,By), d(Ax,Ty)d(By,Ty)\}$
$+c_3 \{d(Sx,By)d(Ty,Ax)\}$

Where $c_1,c_2,c_3 \geq 0$, $c_1+2c_2<1$ and $c_1+c_3> 1$,then A,B,S and T have a unique common fixed point $z\in X$.

1.4 Associated Sequence:
Suppose A, B, S and T are self maps of a metric space (X, d) satisfying the condition (1.3.2), Then for any $x_0\in X$, $Ax_0\in A(X)$ and hence, $Ax_0\in T(X)$ so that there is a $x_1\in X$ with $Ax_0= Tx_1$. Now $Bx_1\in B(X)$ and hence there is $x_2\in X$ with $Bx_1=Sx_2$. Repeating this process to each $x_0\in X$, we get a sequence $<x_n>$ in X such that $Ax_{2n}= Tx_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$. We shall call this sequence as an associated sequence of x_0 relative to the four self maps A, B,S and T.

Now we prove a lemma which plays an important role in proving our theorem.

1.5 Lemma: Suppose A, B, S and T are four self maps of a metric space (X, d) satisfying the conditions (1.3.2) and (1.3.4) of Theorem(1.3) and Further if (1.3.1) (X, d) is a complete metric space then for any $x_0\in X$ and for any of its associated sequence $<x_n>$ relative to Four self maps, the sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots$, converges to some point $z\in X$.

Proof: For simplicity let us take $d_n=d(y_n,y_{n+1})$ for $n=0,1,2,\ldots$

We have

$$d^2_{2n+1}= [d(y_{2n+1},y_{2n+2})]^2= [d(Ax_{2n},Bx_{2n+1})]^2$$
$$\leq c_1\max\{[d(Sx_{2n},Ax_{2n})]^2, d(Tx_{2n+1},Bx_{2n+1})]^2,[d(Sx_{2n},Tx_{2n+1})]\}$$
$$+c_2\max\{d(Sx_{2n},Ax_{2n})d(Sx_{2n},Bx_{2n+1}),\ d(Ax_{2n},Tx_{2n+1}) d(Bx_{2n+1},Tx_{2n+1})\} +c_3 \{d(Sx_{2n}, Bx_{2n+1})\, d(Tx_{2n+1},Ax_{2n})\}$$

$$\leq c_1 \{d^2_{2n},d^2_{2n+1}\}+c_2\{d_{2n}\, d(y_{2n},y_{2n-2})\}$$

$$\leq c_1 \max \{ d^2_{2n},d^2_{2n+1}\}+c_2[d^2_{2n}+d_{2n}d_{2n-1}]$$
$$\leq c_1 \max\{ d^2_{2n},d^2_{2n+1}\}+c_2[\tfrac{3}{2} d^2_{2n}+ \tfrac{1}{2} d^2_{2n-1}] \quad \ldots\ldots(1.5.1)$$

If $d_{2n+1}>d_{2n}$, inequality (1.5.1) implies $d^2_{2n+1} \leq \frac{2c2}{2-2\,c1-c2} d^2_{2n}$ a contradiction, since $\frac{3\,c2}{2-2\,c1-c2}<1$. Thus $d_{2n+1}\leq d_{2n}$ and

inequality (1.5.1) implies that $d_{2n+1}=d(y_{2n+1},y_{2n-2}) \leq h\, d(y_{2n},y_{2n+1})= h^2 d_{2n}$ Where $h^2 = \frac{2c1+3c2}{2-c2} <1$.
Similarly,
$d_{2n}^2=[d(y_{2n},y_{2n+1})]^2=[d(Ax_{2n},Bx_{2n-1})]^2 \leq c_1\max\{d^2_{2n-1},d^2_{2n}\}+c_2(\frac{3}{2} d^2_{2n-1}+\frac{1}{2} d^2_{2n})$ and it follows above that
$d_{2n}=d(y_{2n},y_{2n+1})\leq h\ d\ (y_{2n-1},y_{2n})= d_{2n-1}$

Consequently, $d(y_{n+1},y_n) \leq h\ d(y_n,y_{n-1})$, For n=1,2, 3……since h<1, this implies that $\{y_n\}$ is a cauchy sequence in X.
Hence the Lemma.
The converse of the lemma is not true.
That is, suppose A, B, S and T are self maps of a metric space (X, d) satisfying the conditions (1.3.2) and (1.3.4), even for each associated sequence $<x_n>$ of x_0, the associated sequence converges, the metric space (X,d) need not be complete. For this we provide an example.

Example:

Let X=(-1,1) with d(x,y)= $|x-y|$

$$Ax = Bx = \begin{cases} \frac{1}{5} & if\ -1 < x < \frac{1}{6} \\ \frac{1}{6} & if\ \frac{1}{6} \leq x < 1 \end{cases}$$

$$Sx = Tx = \begin{cases} \frac{1}{5} & if\ -1 < x < \frac{1}{6} \\ \frac{1}{3} - x & if\ \frac{1}{6} \leq x < 1 \end{cases}$$

Then A(X)=B(X)= $\left\{\frac{1}{5},\frac{1}{6}\right\}$ while S(X)=T(X)= $\left\{\frac{1}{5} \cup \left[\frac{1}{6},\frac{-2}{3}\right)\right\}$ so that A(X) $\subset$ T(X) and B(X) $\subset$ S(X).Also the rational inequality holds with $c_1, c_2, c_3 \geq 0$, $c_1+2c_2<1$ and $c_1+c_3> 1$.But (X,d) is not a complete metric space. It is easy to prove that the associated sequence Ax_0, $Bx_1,Ax_2,Bx_3,..,Ax_{2n},Bx_{2n+1}$. converges to $\frac{1}{5}$ if $-1 < x < \frac{1}{6}$; and converges to $\frac{1}{6}$ if $\frac{1}{6} \leq x < 1$.

II MAIN RESULT

Theorem 2: Suppose A, B, S and T are four self maps of metric space(X,d)such that
2.1 $A(x)\subseteq T(x)$, $B(x)\subseteq S(x)$,
2.2 The pairs (A,S) and (B,T) are weakly compatible.
2.3 $d(Ax,By)^2 \leq c_1 \max\{d(Sx,Ax)^2,d(Ty,By)^2,d(Sx,Ty)^2\}$
$+c_2 \max \{d(Sx,Ax)\, d(Sx,By),\, d(Ax,Ty)d(By,Ty)\}$
$+c_3\ d(Sx,By)d(Ty,Ax)\}$
where $c_1,c_2,c_3, \geq 0$, $c_1+2c_2<1$ and $c_1+c_3> 1$
Further if

2.4 The sequence Ax_0, Bx_1, Ax_2, Bx_3….. Ax_2n, Bx_{2n+1}…. converges to $z\in X$ then A ,B,S and T have a unique common fixed point $z\in X$.

Proof: From condition (2.4), Ax_{2n}, Tx_{2n+1}, Bx_{2n+1}, Sx_{2n} converges to z as $n \to \infty$.
Also Since $A(x)\subseteq T(x)$ $\exists$ $u\in x$ such that z=Tu
To prove Bu=z, consider

$d(Ax_{2n},Bu)^2 < c_1 \max\{d(Sx_{2n},Ax_{2n})^2,d(Tu,Bu)^2,d(Sx_{2n},Tu)^2\}$
$+c_2 \max \{d(Sx_{2n},Ax_{2n})\, d(Sx_{2n},Bu),\, d(Ax_{2n},Tu)d(Bu,Tu)\}$
$+c_3 \{d(Sx_{2n},Bu)d(Tu,Ax_{2n})\}$

Letting n tends to ∞, we get
$d[(z.Bu)]^2 \leq c_1 \max\{[d(z,z)^2,d(z,Bu)^2,d(z,z)^2]\}$
$+c_2 \max \{d(z,z)\, d(z,Bu),\, d(z,z)d(Bu,z)\}$
$+ c_3 \{d(z,Bu)d(z,z)\}$
$= c_1\, d(z,Bu)^2$
$d(z,Bu)^2[1- c_1]\leq 0$. Since c_1 cannot be greater than 1, d(z,Bu)=0, this gives Bu=z.
Since the pair (B,T) is weakly compatible and z= Bu=Tu, d(BBu, TTu)=0 or Bz=Tz.

To prove Bz=z put $x=x_{2n}$, y=z in (2.3)
$d(Ax_{2n},Bz)^2 < c_1 \max\{d(Sx_{2n},Ax_{2n})^2,d(Tz,Bz)^2,d(Sx_{2n},Tz)^2\}$
$+c_2 \max \{d(Sx_{2n},Ax_{2n})\, d(Sx_{2n},Bz),\, d(Ax_{2n},Tz)d(Bz,Tz)\}$
$+c_3 \{d(Sx_{2n},Bz)d(Tz,Ax_{2n})\}$
Letting n tends to ∞, using Bz=Tz. we get

$d[(z.Bz)]^2 \leq c_1 \max\{[d(z,z)^2,d(Bz,Bz)^2,d(z,Bz)^2]\}$
$+c_2 \max \{d(z,z)\, d(z,Bz),\, d(z,Bz)d(Bz,Bz)\}$
$+ c_3 \{d(z,Bz)d(Bz,z)\}$
$d[(z.Bz)]^2 \leq c_1\, d(z,Bz)^2 + c_3\, d(z,Bz)^2$
$d(z,Bz)^2[1- (c_1+c_3)]\leq 0$, since c_1+c_3 cannot be greater than 1, d(z,Bz)=0, this gives Bz=z. Therefore z=Bz=Tz.

Also Since $B(x)\subseteq S(x)$ $\exists$ $v\in x$ such that z=Sv
To prove Av=z, put x=v, y=z in (2.3) we get

$d(Av,Bz)^2 \leq c_1 \max\{[d(Sv,Av)^2,d(Tz,Bz)^2,d(Sv,Tz)^2]\}$
$+c_2 \max \{d(Sv,Av),\, d(Sv,Bz),\, d(Av,Tz),d(Bz,Tz)\}$
$+c_3 \{d(Sv,Bz),d(Tz,Av)\}$

$d(Av,Bz)^2 \leq c_1 \max\{[d(z,Av)^2,d(Bz,Bz)^2,d(z,Tz)^2]\}$
$+c_2 \max \{d(z,Av),\, d(z,Bz),\, d(Av,Tz),d(Tz,Tz)\}$
$+c_3 \{d(z,Bz),d(Tz,Av)\}$

$d(Av,z)^2 \leq c_1 \max\{[d(z,Av)^2,d(z,z)^2,d(z,z)^2]\}$
$+c_2 \max \{d(z,Av),\, d(z,z),\, d(Av,z),d(z,z)\}$
$+c_3 \{d(z,z),d(z,Av)\}$

$d(Av,z)^2 \leq c_1\, d(z,Av)^2$
$d(Av,z)^2[1-c_1] \leq 0$. Since c_1 cannot be greater than 1, d(Av,z)=0, this gives Av=z. Hence Av=Sv=z.

Since the pair (A,S) is weakly compatible ASv=SAv or Az=Sz.

To prove Az=z, consider

$d(Az,Bz)^2 \leq c_1 \max\{[d(Sz,Az)^2,d(Tz,Bz)^2,d(Sz,Tz)^2]\}$
$+c_2 \max \{d(Sz,Az)\, d(Sz,Bz),\, d(Az,Tz)d(Bz,Tz)\}$
$+c_3 \{d(Sz,Bz)d(Tz,Az)\}d(Az,z)^2\}$

$\leq c_1 d(Az,z)^2 + c_3 d(z,Az)^2$.

This gives

$d(Az,z)^2 \leq (c_1+c_3)\ d(z,Bz)^2$

$d(Az,z)^2[1-(c_1+c_3)] \leq 0$, since $c_1+c_3<1$, we get $d(Az,z)^2=0$ or z=Az . Therefore S z=Az=z. Since z=Az=Bz=Sz=Tz, z is a common fixed point of A, B, S and T.

The uniqueness of common fixed point can be easily proved.

Now, we discuss our earlier example in the following two remarks to justify our result.

Remark-1: The pairs (A,S) and (B,T) are weakly compatible as they commute at coincident points $\frac{1}{5}$ and $\frac{1}{6}$ More over $\frac{1}{6}$ is the unique common fixed point of P,Q,S and T.

Remark-2: In view of the earlier example Theorem2 is the generalization of theorem 1by virtue of weaker conditions such as weakly compatible mappings in place of compatible mappings; Associated sequence in place of completeness of metric space(X,d) and the continuity condition is being dropped.

11. REFERENCES

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