Anti-fuzzy KUS-ideals of KUS-algebras

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ABSTRACT
In this paper , we introduce the notion of anti-fuzzy KUS-ideals in KUS-algebra, several appropriate examples are provided and their some properties are investigated . The image and the inverse image of anti-fuzzy KUS-ideals in KUS-algebra are defined and how the image and the inverse image of anti-fuzzy KUS-ideals in KUS-algebra become anti-fuzzy KUS-ideals are studied . Moreover , the cartesian product of anti-fuzzy KUS-ideals are given .

Keywords
KUS-ideals , fuzzy KUS-ideals ,anti-fuzzy KUS-ideals , image and pre-image of anti-fuzzy KUS-ideals.

1. INTRODUCTION
After the introduction of fuzzy subsets by L.A. Zadeh [10], several researchers explored on the generalization of the notion of fuzzy subset. H.V. Kumbhojkar and M.S. Bapat [4] defined not-so-fuzzy fuzzy ideals, N. Palaniappan and K. Arjunan [7] defined the anti-homomorphism of a fuzzy and an anti-fuzzy ideals.BCK - algebras form an important class of logical algebras introduced by K. Iseki [2] and was extensively investigated by several researchers. The class of all BCK-algebras is quasi variety. Y. B. Jun , J. Meng and et al posed an interesting problem (solved in ([3], [5])) whether the class of all BCK-algebras is a variety. In ([8],[9]) , C. Prabpayak and U. Leerawat introduced a new algebraic structure ,which is called KU-algebra . They gave the concept of homomorphisms of KU-algebras and investigated some related properties. S. Mostafa ,and et al (in [6]) introduced the notion of fuzzy KUS-ideals of KUS-algebras and they investigated several basic properties which are related to fuzzy KUS-ideals. they described how to deal with the homomorphism image and inverse image of fuzzy KUS-ideals. they have also proved that the cartesian product of fuzzy KUS-ideals in cartesian product of fuzzy KUS-algebras are fuzzy KUS-ideals. In this paper, we introduce the notion of anti-fuzzy KUS-ideals of KUS-algebras and then we study the homomorphism image and inverse image of anti-fuzzy KUS-ideals. We also prove that the cartesian product of anti-fuzzy KUS-ideals is an anti-fuzzy KUS-ideals .

2. Preliminaries
In this section we give some basic definitions and preliminaries lemmas of KUS-ideals and fuzzy KUS-ideals of KUS-algebras .

Definition 2.1([6]). Let $(X; *,0)$ be an algebra with a single binary operation $(*)$. X is called a KUS-algebra if it satisfies the following identities: for any $x, y, z \in X$,

(kus_1) : $(z * y) * (z * x) = y * x$,
(kus_2) : $0 * x = x$,
(kus_3) : $x * x = 0$,
(kus_4) : $x * (y * z) = y * (x * z)$.

In what follows, let $(X; *,0)$ denote a KUS-algebra unless otherwise specified.
For brevity we also call X a KUS-algebra. In X we can define a binary relation $(\leq)$
by : $x \leq y$ if and only if $y * x = 0$.

Lemma 2.2 ([6]). In any KUS-algebra $(X;*,0)$, the following properties holds: for all $x, y, z \in X$;

a) $x * y = 0$ and $y * x = 0$ imply $x = y$,
b) $y * [(y * z) * z] = 0$,
c) $(0 * x) * (y * x) = y * 0$,
d) $x \leq y$ implies that $y * z \leq x * z$ and $z * x \leq z * y$,
e) $x \leq y$ and $y \leq z$ imply $x \leq z$,
f) $x * y \leq z$ implies that $z * y \leq x$.

Definition 2.3 ([6]). A nonempty subset S of a KUS-algebra X is called a KUS-sub-algebra of X if $x * y \in S$, whenever $x , y \in S$.

Definition 2.4 ([6]). A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for $x , y, z \in X$,
$(Ikus_1)$ $(0 \in I)$,
$(Ikus_2)$ $(z * y) \in I$ and $(y * x) \in I$ imply $(z * x) \in I$.

Definition 2.5 ([9]). Let $(X ; *,0)$ and $(Y; *`,0`)$ be nonempty sets . The mapping $f : (X; *,0) \to (Y; *`,0`)$ be called a homomorphism if it satisfying $f(x * y) = f(x) *` f(y)$ for all $x , y \in X$. The set $\{x \in X \mid f(x) = 0'\}$ is called the Kernel of f denoted by Ker f .

Definition 2.6([10]). Let $(X; *,0)$ be a nonempty set, a fuzzy subset μ in X is a function $\mu: X \to [0,1]$.

Definition 2.7([6]). Let $(X; *,0)$ be a KUS-algebra , a fuzzy subset μ in X is called a fuzzy KUS-sub-algebra of X if for all $x , y \in X$,
$\mu (x * y) \geq \min \{\mu (x), \mu (y)\}$.

Definition 2.8([6]). Let $(X; *,0)$ be a KUS-algebra , a fuzzy subset μ in X is called a fuzzy KUS-ideal of X if it satisfies the following conditions: , for all $x , y, z \in X$,
$(Fkus_1)$ $\mu (0) \geq \mu (x)$,
$(Fkus_2)$ $\mu (z * x) \geq \min \{\mu (z * y), \mu (y * x)\}$.

3. Anti-fuzzy KUS-ideals of

KUS-algebras

In this section, we will introduce a new notion called an anti-fuzzy KUS-ideal of KUS-algebra and study several basic properties of it.

Definition 3.1. Let $(X; *, 0)$ be a KUS-algebra. A fuzzy set μ in X is called an anti-fuzzy KUS-ideal of X if it satisfies the following conditions: for all x, y and z $\in$ X,
($Akus_1$) $\mu(0) \leq \mu(x)$.
($Akus_2$) $\mu(z * x) \leq \max\{\mu(z * y), \mu(y * x)\}$.

Example 3.2. Let X = {0, 1, 2, 3} in which ($*$) is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Then $(X; *, 0)$ is a KUS-algebra. Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = t_1$,
$\mu(1) = \mu(2) = \mu(3) = t_2$, where $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Routine calculation gives that μ is an anti-fuzzy KUS-ideal of KUS-algebras X.

Lemma 3.3. Let μ be an anti-fuzzy KUS-ideal of KUS-algebra X and if $x \leq y$, then $\mu(x) \leq \mu(y)$, for all x, y $\in$ X.
Proof: Assume that $x \leq y$, then $y * x = 0$, and
$\mu(0 * x) = \mu(x) \leq \max\{\mu(0 * y), \mu(y * x)\}$
$= \max\{\mu(y), \mu(0)\} = \mu(y)$. Hence $\mu(x) \leq \mu(y)$. ⌂

Proposition 3.4. Let μ be an anti-fuzzy KUS-ideal of KUS-algebra X. If the inequality $y * x \leq z$ hold in X, then $\mu(x) \leq \max\{\mu(y), \mu(z)\}$.
Proof: Assume that the inequality $y * x \leq z$ hold in X, by lemma (3.3), $\mu(y * x) \leq \mu(z)$ --- (1).
By($Akus_2$), $\mu(z * x) \leq \max\{\mu(z * y), \mu(y * x)\}$. Put z = 0, then
$\mu(0 * x) = \mu(x) \leq \max\{\mu(0*y), \mu(y * x)\}$
$= \max\{\mu(y), \mu(y * x)\}$ --- (2) .
From (1) and (2), we get $\mu(x) \leq \max\{\mu(y), \mu(z)\}$, for all x, y, z $\in$X. ⌂

Theorem 3.5. Let μ be an anti-fuzzy set in X then μ is an anti-fuzzy KUS-ideal of X if and only if it satisfies: if $U(\mu,\alpha) = \{x \in X \mid \mu(x) \leq \alpha\}$, for all $\alpha \in [0, 1]$, $U(\mu,\alpha) \neq \varnothing$ implies $U(\mu,\alpha)$ is a KUS-ideal of X---- (A) .
Proof: Assume that μ is an anti-fuzzy KUS-ideal of X, let $\alpha \in [0,1]$ be such that $U(\mu,\alpha) \neq \varnothing$, and let x, y $\in$ X be such that $x \in U(\mu,\alpha)$, then $\mu(x) \leq \alpha$ and so by ($Akus_1$), $\mu(0) \leq \mu(x) \leq \alpha$. Thus
$0 \in U(\mu,\alpha)$.
Now let $(z * y), (y * x) \in U(\mu,\alpha)$. It follows from ($Akus_2$) that
$\mu(z * x) \leq \max\{\mu(z * y), \mu(y * x)\} = \alpha$, so that $(z * x) \in U(\mu,\alpha)$. Hence $U(\mu,\alpha)$ is a KUS-ideal of X.
Conversely, suppose that μ satisfies (A), assume that ($Akus_1$) is false, then there exist $x \in X$ such that $\mu(0) > \mu(x)$. If we take $t = \frac{1}{2}[\mu(x)+\mu(0)]$, then $\mu(0) > t$ and $0 \leq \mu(x) < t \leq 1$, thus $x \in U(\mu, t)$ and $U(\mu,t) \neq \varnothing$. As $U(\mu, t)$ is a KUS-ideal of X, we have $0 \in U(\mu, t)$, and so $\mu(0) \leq t$. This is a contradiction. Hence $\mu(0) \leq \mu(x)$ for all $x \in X$. Now, assume ($Akus_2$) is not true then there exist
x, y, z $\in$ X such that
$\mu(z * x) > \max\{\mu(z * y), \mu(y * x)\}$,
taking $\beta_0 = \frac{1}{2}[\mu(z * x)+\max\{\mu(z * y), \mu(y * x)\}]$, we have $\beta_0 \in [0, 1]$ and
$\max\{\mu(z * y), \mu(y * x)\} < \beta_0 < \mu(z * x)$, it follows that $\max\{\mu(z * y), \mu(y * x)\} \in U(\mu,\beta_0)$ and
$z * x \notin U(\mu,\beta_0)$, this is a contradiction and therefore μ is an anti-fuzzy KUS-ideal of X. ⌂

4. Characterization of anti-fuzzy KUS-ideals by their level KUS-ideals

Theorem 4.1. A fuzzy subset μ of a KUS-algebra X is an anti-fuzzy KUS-ideal of X if and only if, for every $t \in [0,1]$, μ_t is either empty or a KUS-ideal of X, where $\mu_t = \{x \in X \mid \mu(x) \leq t\}$.
Proof: Assume that μ is an anti-fuzzy KUS-ideal of X, by ($Akus_1$), we have $\mu(0) \leq \mu(x)$ for all $x \in X$, therefore $\mu(0) \leq \mu(x) \leq t$, for $x \in \mu_t$ and so
$0 \in \mu_t$.
Let $(z * y) \in \mu_t$ and $(y * x) \in \mu_t$, then $\mu(z*y) \leq t$ and $\mu(y * x) \leq t$, since μ is an anti-fuzzy KUS-ideal it follows that
$\mu(z * x) \leq \max\{\mu(z * y), \mu(y * x)\} \leq t$ and that $(z * x) \in \mu_t$. Hence μ_t is a KUS-ideal of X.
Conversely, we only need to show that ($Akus_1$) and ($Akus_2$) are true. If ($Akus_1$) is false, then there exist $x \in X$ such that $\mu(0) > \mu(x)$. If we take
$t = \frac{1}{2}(\mu(x) + \mu(0))$, then $\mu(0) > t$ and
$0 \leq \mu(x) < t \leq 1$ thus $x \in \mu_t$ and $\mu_t \neq \varnothing$. As μ_t is a KUS-ideal of X, we have $0 \in \mu_t$ and so $\mu(0) \leq t$. This is a contradiction.
Now, assume ($Akus_2$) is not true, then there exist x, y and z $\in$ X such that,
$\mu(z*x) > \max\{\mu(z*y), \mu(y*x)\}$.
Putting $t = \frac{1}{2}[\mu(z * x)+ \max\{\mu(z * y), \mu(y * x)\}]$, then $\mu(z * x) > t$ and
$0 \leq \max\{\mu(z * y),\mu(y * x)\} < t \leq 1$, hence
$\mu(z * y) < t$ and $\mu(y * x) < t$, which imply that
$(z * y) \in \mu_t$ and $(y * x) \in \mu_t$, since μ_t is an anti-fuzzy KUS-ideal, it follows that $(z * x) \in \mu_t$ and that $\mu(z * x) \leq t$, this is also a contradiction. Hence μ is an anti-fuzzy KUS-ideal of X . ⌂

Corollary 4.2. If a fuzzy subset μ of KUS-algebra X is an anti-fuzzy KUS-ideal, then for every

$t \in Im(\mu)$, μ_t is a KUS-ideal of X.

Remark 4.3. Let μ be an anti-fuzzy KUS-ideal of KUS-algebra X, then the KUS-ideal μ_t, $t \in [0,1]$ are called level KUS-ideals of μ.

Corollary 4.4. Let I be a KUS-ideal of a KUS-algebra X, then for any fixed number t in an open interval (0,1), there exist an anti-fuzzy KUS-ideal μ of X such that μ_t =I.

Proof: Define $\mu : X \to [0:1]$ by

$$\mu(x) = \begin{cases} 0, & \text{if } x \in I; \\ t, & \text{if } x \notin I. \end{cases}$$

Where t is a fixed number in (0,1). Clearly, $\mu(0) \le \mu(x)$ and we have one two level sets $\mu_0 = I$, $\mu_t = X$, which are KUS-ideals of X, then from Theorem (4.1) μ is an anti-fuzzy KUS-ideal of X. ⌂

5. Image and Pre-image of anti-fuzzy KUS-ideals

Definition 5.1([2]). $f : (X; *, 0) \to (Y; *`, 0`)$ be a mapping from a nonempty set X to a nonempty set Y. If β is a fuzzy subset of X, then the fuzzy subset μ of Y defined by:

$$f(\mu)(y) = \beta(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f.

Similarly if μ is a fuzzy subset of Y, then the fuzzy subset $\mu = (\beta \circ f)$ in X (i.e., the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$) is called the pre-image of β under f.

Theorem 5.2. An onto homomorphic pre-image of anti-fuzzy KUS-ideal is also an anti-fuzzy KUS-ideal.

Proof: Let $f : (X; *, 0) \to (Y; *`, 0`)$ be an onto homomorphism of KUS-algebras,
β is an anti-fuzzy KUS-ideal of Y and μ the pre-image of β under f, then

$\beta(f(x)) = \mu(x)$, for all $x \in X$. Let $x \in X$, then

$\mu(0) = \beta(f(0)) < \beta(f(x)) = \mu(x)$. Now let

x, y, z $\in$ X, then

$\mu(z * x) = \beta(f(z * x)) = \beta(f(z) *' f(x))$

$\le \max\{\beta(f(z) *' f(y)), \beta(f(y) *' f(x))\}$

$= \max\{\beta(f(z * y)), \beta(f(y * x))\}$

$= \max\{\mu(z * y), \mu(y * x)\}$, and the proof is completed. ⌂

Definition 5.3. An anti fuzzy subset μ of X has inf property if for any subset T of X, there exist $t_0 \in T$ such that $\mu(t) = \inf_{t \in T} \mu(t)$.

Theorem 5.4. Let $f : (X; *, 0) \to (Y; *`, 0`)$ be an onto homomorphism between KUS-algebras X and Y respectively. For every anti-fuzzy KUS-ideal μ in X, $f(\mu)$ is an anti-fuzzy KUS-ideal of Y.

Proof: Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) \ge \mu(f(0)) = \mu(0')$.

Let $f : X \to Y$ be an onto homomorphism of KUS-algebras, μ is an anti-fuzzy KUS-ideal of X with inf property and β the image of μ under f, since μ is anti-fuzzy KUS-ideal of X, we have
$\mu(0) \le \mu(x)$ for all $x \in X$. Note that $0 \in f^{-1}(0')$, where 0, 0' are the zero of X and Y, respectively. Thus $\beta(0') = \inf_{t \in f^{-1}(x')} \mu(t)$
$= \beta(x')$, for all $x \in X$, which implies that
$\beta(0') \le \inf_{t \in f^{-1}(x')} \mu(t) = \beta(x')$, for any $x' \in Y$. Let x', y', z' $\in$ Y, then there exists $x_0, y_0, z_0 \in X$ such that $x_0 = f^{-1}(x')$, $y_0 = f^{-1}(y')$, $z_0 = f^{-1}(z')$. It follows that

$\mu(z_0 * y_0) = \inf_{t \in f^{-1}(z' * y')} \mu(t)$,

$\mu(y_0 * x_0) = \inf_{t \in f^{-1}(y' * x')} \mu(t)$ and

$\mu(z_0 * x_0) = \inf_{t \in f^{-1}(z' * x')} \mu(t)$. Then

$f(\mu)(z' * x') = \beta(z' * x') = \inf_{t \in f^{-1}(z' * x')} \mu(t)$

$= \mu(z_0 * x_0) \le \max\{\mu(z_0 * y_0), \mu(y_0 * x_0)\}$

$= \max[\inf_{t \in f^{-1}(z' * y')} \mu(t), \inf_{t \in f^{-1}(y' * x')} \mu(t)]$

$= \max\{\beta(z' * y'), \beta(y' * x')\}$.

$= \max\{f(\mu)(z' * y'), f(\mu)(y' * x')\}$.

Hence $f(\mu)$ is an anti-fuzzy KUS-ideal of Y. ⌂

6. Cartesian product of anti-fuzzy KUS-ideals

Definition 6.1 ([1]). A fuzzy relation R on any set S is a fuzzy subset R: $S \times S \to [0,1]$.

Definition 6.2 ([1]). If R is a fuzzy relation on sets S and β is a fuzzy subset of S, then R is a fuzzy relation on β if $R(x, y) \ge \max\{\beta(x), \beta(y)\}$, for all $x, y \in S$.

Definition 6.3([1]). Let μ and β be fuzzy subsets of a set S. The cartesian product of μ and β is defined by $(\mu \times \beta)(x,y) = \max\{\mu(x), \beta(y)\}$, for all $x, y \in S$.

Lemma 6.4([1]). Let S be a set and μ and β be fuzzy subsets of S. Then,

(1) $\mu \times \beta$ is a fuzzy relation on S,

(2) $(\mu \times \beta)_t = \mu_t \times \beta_t$, for all $t \in [0,1]$.

Definition 6.5([1]). Let S be a set and β be fuzzy subset of S. The strongest fuzzy relation on S, that is, a fuzzy relation on β is R_β given by

$R_\beta(x,y) = \max\{\beta(x), \beta(y)\}$, for all $x, y \in S$.

Lemma 6.6([1]). For a given fuzzy subset β of a set S, let R_β be the strongest fuzzy relation on S. Then for $t \in [0,1]$, we have $(R_\beta)_t = \beta_t \times \beta_t$.

Proposition 6.7. For a given fuzzy subset β of a KUS-algebra X, let R_β be the strongest fuzzy relation on X. If β is an anti-fuzzy KUS-ideal of $X \times X$, then $R_\beta(x,x) \ge R_\beta(0,0)$, for all $x \in X$

Proof: Since R_β is a strongest fuzzy relation of $X\times X$, it follows from that,
$R_\beta(x,x) = \max\{\beta(x), \beta(x)\} \geq \max\{\beta(0), \beta(0)\}$
$= R_\beta(0,0)$, which implies that $R_\beta(x,x) \geq R_\beta(0,0)$. ⌂

Proposition 6.8. For a given fuzzy subset β of a KUS-algebra X, let R_β be the strongest fuzzy relation on X. If R_β is an anti-fuzzy KUS-ideal of $X\times X$, then $\beta(x) \geq \beta(0)$, for all $x\in X$.
Proof: Since R_β is an anti-fuzzy KUS-ideal of $X\times X$, it follows from ($Akus_1$),
$R_\beta(x,x) \geq R_\beta(0,0)$, where $(0,0)$ is the zero element of $X\times X$. But this means that, $\max\{\beta(x), \beta(x)\} \geq \max\{\beta(0), \beta(0)\}$ which implies that $\beta(x) \geq \beta(0)$. ⌂

Remark 6.9([1]). Let X and Y be KUS-algebras, we define (*) on $X\times Y$ by : for all $(x,y),(u,v) \in X\times Y$, $(x,y) * (u,v) = (x * u, y * v)$. Then clearly $(X\times Y; * .(0,0))$ is a KUS-algebra.

Theorem 6.10. Let μ and β be an anti-fuzzy KUS-ideals of KUS-algebra X . Then $\mu\times\beta$ is an anti-fuzzy KUS-ideal of $X\times X$.
Proof: Note first that for every $(x,y)\in X\times X$,
$(\mu\times\beta)(0,0) = \max\{\mu(0), \beta(0)\}$
$\leq \max\{\mu(x), \beta(y)\} = (\mu\times\beta)(x,y)$.

Now let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$. Then
$(\mu\times\beta)(z_1 * x_1, z_2 * x_2) = \max\{\mu(z_1 * x_1), \beta(z_2 * x_2)\} \leq \max\{\max\{\mu(z_1 * y_1), \mu(y_1 * x_1)\},$
$\max\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}\}$
$= \max\{\max\{\mu(z_1 * y_1), \beta(z_2 * y_2)\},$
$\max\{\mu(y_1 * x_1), \beta(y_2 * x_2)\}\}$
$= \max\{(\mu\times\beta)(z_1 * y_1, z_2 * y_2),$
$(\mu\times\beta)(y_1 * x_1, y_2 * x_2)\}$
Hence $(\mu\times\beta)$ is an anti-fuzzy KUS-ideal of $X\times X$.⌂

Theorem 6.11. Let μ and β be anti-fuzzy subsets of KUS-algebra X such that $\mu\times\beta$ is an anti-fuzzy KUS-ideal of $X\times X$. Then for all $x\in X$,
(i) either $\mu(0) \leq \mu(x)$ or $\beta(0) \leq \beta(x)$.
(ii) $\mu(0) \leq \mu(x)$,then either $\beta(0) \leq \beta(x)$ or $\beta(0) \leq \mu(x)$.
(iii) If $\beta(0) \leq \beta(x)$, then either $\mu(0) \leq \mu(x)$ or $\beta(0) \leq \mu(x)$.
(iv) Either μ or β is an anti-fuzzy KUS-ideal of X .
Proof.
(i) suppose that $\mu(0) > \mu(x)$ and $\beta(0) > \beta(y)$ for some $x, y \in X$. Then $(\mu\times\beta)(x,y)$
$= \max\{\mu(x), \beta(y)\} < \max\{\mu(0), \beta(0)\} = (\mu\times\beta)(0,0)$. This is a contradiction and we obtain (i).
(ii) Assume that there exist $x, y \in X$ such that $\beta(0) > \mu(x)$ and $\beta(0) > \beta(y)$. Then $(\mu\times\beta)(0,0) = \max\{\mu(0),\beta(0)\} = \beta(0)$ it follows that $(\mu\times\beta)(x,y) = \max\{\mu(x), \beta(y)\} < \beta(0) = (\mu\times\beta)(0,0)$ which is a contradiction. Hence (ii) holds.
(iii) is by similar method to part (ii).
(iv) Suppose $\beta(0) \leq \beta(x)$ by (i), then form (iii) either $\mu(0) \leq \mu(x)$ or $\beta(0) \leq \mu(x)$ for all $x\in X$.

If $\mu(0) \leq \beta(x)$, for any $x\in X$, then $(\mu\times\beta)(0,x) = \max\{\mu(0), \beta(x)\} = \beta(x)$. Let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$, since $(\mu\times\beta)$ is an anti-fuzzy KUS-ideal of $X\times X$, we have
$(\mu\times\beta)(z_1 * x_1, z_2 * x_2) \leq \max\{(\mu\times\beta)(z_1 * y_1, z_2 * y_2),$
$(\mu\times\beta)(y_1 * x_1, y_2 * x_2)\}$---- (A)
If we take $x_1 = y_1 = z_1 = 0$, then
$\beta(z_2 * x_2) = (\mu\times\beta)(0, z_2 * x_2)$
$\leq \max\{(\mu\times\beta)(0, z_2 * y_2), (\mu\times\beta)(0, y_2 * x_2)\}$
$= \max\{\max\{\mu(0), \beta(z_2 * y_2)\},$
$\max\{\mu(0), \beta(y_2 * x_2)\}\}$
$= \max\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}$
This prove that β is an anti-fuzzy KUS-ideal of X .
Now we consider the case $\mu(0) \leq \mu(x)$ for all $x\in X$. Suppose that $\mu(0) > \mu(y)$ for some $y \in X$. then
$\beta(0) \leq \beta(y) < \mu(0)$.
Since $\mu(0) \leq \mu(x)$ for all $x\in X$, it follows that $\beta(0) < \mu(x)$ for any $x\in X$.
Hence $(\mu\times\beta)(x,0) = \max\{\mu(x), \beta(0)\} = \mu(x)$ taking $x_2 = y_2 = z_2 = 0$ in (A), then
$\mu(z_1 * x_1) = (\mu\times\beta)(z_1 * x_1, 0)$
$\leq \max\{(\mu\times\beta)(z_1 * y_1, 0), (\mu\times\beta)(y_1 * x_1, 0)\}$
$= \max\{\max\{\mu(z_1 * y_1), \beta(0)\},$
$\max\{\mu(y_1 * x_1), \beta(0)\}\}$
$= \max\{\mu(z_1 * y_1), \mu(y_1 * x_1)\}$
Which proves that μ is an anti-fuzzy KUS-ideal of X .
Hence either μ or β is an anti-fuzzy KUS-ideal of X . ⌂

Theorem 6.12. Let β be a fuzzy subset of a KUS-algebra X and let R_β be the strongest fuzzy relation on X , then β is an anti-fuzzy KUS-ideal of X if and only if R_β is an anti-fuzzy KUS-ideal of $X\times X$.
Proof: Assume that β is an anti-fuzzy KUS-ideal of X . By proposition (6.7) ,we get,
$R_\beta(0,0) \leq R_\beta(x,y)$, for any $(x,y)\in X\times X$.
Let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$, we have from ($Akus_2$) :
$R_\beta(z_1 * x_1, z_2 * x_2) = \max\{\beta(z_1 * x_1), \beta(z_2 * x_2)\}$
$\leq \max\{\max\{\beta(z_1 * y_1), \beta(y_1 * x_1)\},$
$\max\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}\}$
$= \max\{\max\{\beta(z_1 * y_1), \beta(z_2 * y_2)\},$
$\max\{\beta(y_1 * x_1), \beta(y_2 * x_2)\}\}$
$= \max\{R_\beta(z_1 * y_1, z_2 * y_2), R_\beta(y_1 * x_1, y_2 * x_2)\}$
Hence R_β is an anti-fuzzy KUS-ideal of $X\times X$.
Conversely, suppose that R_β is an anti-fuzzy KUS-ideal of $X\times X$, by proposition (6.8) $\beta(0) \leq \beta(x)$ for all $x \in X$, which prove ($Akus_1$) .
Now, let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$. Then,
$\max\{\beta(z_1 * x_1), \beta(z_2 * x_2)\} = R_\beta(z_1 * x_1, z_2 * x_2)$
$\leq \max\{R_\beta((z_1,z_2) * (y_1, y_2)), R_\beta((y_1,y_2) * (x_1, x_2))\}$
$= \max\{R_\beta((z_1 * y_1),(z_2 * y_2)),$
$R_\beta((y_1 * x_1),(y_2 * x_2))\}$
$= \max\{\max\{\beta(z_1 * y_1), \beta(z_2 * y_2)\},$
$\max\{\beta(y_1 * x_1), \beta(y_2 * x_2)\}\}$
In particular if we take $x_2 = y_2 = z_2 = 0$, then
$\beta(z_1 * x_1) \leq \max\{\beta(z_1 * y_1), \beta(y_1 * x_1)\}$. This proves ($Akus_2$) and β is an anti- fuzzy KUS-ideal of X . ⌂

Theorem 6.13. Let μ and β be fuzzy subsets of a KUS-algebra X such that $\mu\times\beta$ is an anti-fuzzy KUS-ideal of $X\times X$. Then μ or β is an anti-fuzzy KUS-ideal of X.

Proof: By theorem (6.11(i)), without loss of generality we assume that $\mu(x) \geq \mu(0)$ for all $x \in X$. From theorem (6.11(iii)), it follows that either
$\beta(0) \leq \beta(x)$ or $\beta(0) \leq \mu(x)$. If $\mu(x) \geq \beta(0)$ for all $x \in X$, then $(\mu\times\beta)(0,x) = \max\{\beta(0), \mu(x)\} =$
$\mu(x)$. Let $(x,y) \in X\times X$, since $\mu\times\beta$ is an anti-fuzzy KUS-ideal of X. By proposition (6.7), we get, $(\mu\times\beta)(0,0)$
$\leq (\mu\times\beta)(x,y)$.

Let (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X\times X$, we have from $(Akus_2)$:

$(\mu\times\beta)(z_1 * x_1, z_2 * x_2) = \max\{\mu(z_1 * x_1), \beta(z_2 * x_2)\}$
$\leq \max\{\max\{\mu(z_1 * y_1), \mu(y_1 * x_1)\},$
$\max\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}\}$
$= \max\{\max\{\mu(z_1 * y_1), \beta(z_2 * y_2)\},$
$\max\{\mu(y_1 * x_1), \beta(y_2 * x_2)\}\}$
$= \max\{(\mu\times\beta)(z_1 * y_1, z_2 * y_2),$
$(\mu\times\beta)(y_1 * x_1, y_2 * x_2)\}$

In particular if we take $x_1 = y_1 = z_1 = 0$, then
$\beta(z_2 * x_2) = (\mu\times\beta)(0, z_2 * x_2) \leq$
$\max\{(\mu\times\beta)(0,(z_2 * y_2)), (\mu\times\beta)(0,(y_2 * x_2))\}$
$= \max\{\max\{\mu(0), \beta(z_2 * y_2)\},$
$\max\{\mu(0), \beta(y_2 * x_2)\}\}$
$= \max\{\beta(z_2 * y_2), \beta(y_2 * x_2)\}$. This proves that β is an anti-fuzzy KUS-ideal of X. The second part is similar .This completes the proof . ⌂

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