Pair Sum Labeling of some Special Graphs

K.Manimekalai
Department of Mathematics,
Bharathi Women's College (Autonomous),
Chennai -600106, India.

K.Thirusangu
Department of Mathematics,
S.I.V.E.T.College, Gowrivakkam,
Chennai -600073, India.

ABSTRACT

Let G be a (p, q) graph. A one-one map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function, $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$ according as q is even or odd. Recently, the pair sum labeling was introduced by R.Ponraj, J. V. X. Parthipan [3]. In this paper we study about the pair sum labeling of the coconut tree CT(m, n), the Y-tree Y_{n+1}, the Jelly fish graph J(m, n), the (m, 2)-kite, (m, 1)-kite, the theta graph $\Theta(l^{[m]})$, for m even and complete binary tree.

Keywords

pair sum labeling, pair sum graph.

AMS Classification: 05C78.

1. INTRODUCTION

The graph considered here are all finite, undirected and simple. V(G) and E(G) denote the vertex set and edge set of a graph G. The pair sum labeling is introduced in [3] by R. Ponraj and et al. In [3],[4] [5] and [6] they study the pair sum labeling of cycle, path, star and some of their related graphs like $B_{m,n}$, $K_{1,n} \cup K_{1,m}$, $P_m \cup K_{1,n}$, $C_n \cup C_n$ etc. In this paper we prove that the pair sum labeling of coconut tree CT(m, n), the Y-tree Y_{n+1}, Jelly fish J(m, n) , (m, 2)-kite, (m, 1)-kite and the theta graph $\Theta(l^{[m]})$, for m even and complete binary tree. Let x be any real number, then $[x]$ denotes the largest integer less than or equal to x. Terms and terminology as in Harary [2].

Definition 1.1[3]: Let G(V, E) be a (p, q) graph. A one-one function $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{q/2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph.

Definition 1.2: A coconut Tree CT(m,n) is the graph obtained from the path P_m by appending n new pendent edges at an end vertex of P_m .

Definition 1.3: A Y -tree $Y_{n+1.}$ is a graph obtained from the path P_n by appending an edge to a vertex of the path P_m adjacent to an end point.

Definition 1.4: An (n, t)-kite is a cycle of length n with a t-edge path (the tail) attached to one vertex. In particular, the (n, 1)-kite is a cycle of length n with an edge attached to one vertex. (n, 1)-kite is also known as flag $F\ell_n$.

Definition 1.5: The Jelly fish graph J(m, n) is obtained from a 4- cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

Definition 1.6: Take k paths of length $l_1, l_2, l_3, \ldots, l_k$, where $k \geq 3$ and $l_i = 1$ for at most one i. Identify their end points to form a new graph. The new graph is called a generalized theta graph, and is denoted by $\Theta(l_1, l_2, l_3, \ldots, l_k)$. In other words, $\Theta(l_1, l_2, l_3, \ldots, l_k)$ consists $k \geq 3$ pair wise internally disjoint paths of length $l_1, l_2, l_3, \ldots, l_k$ that share a pair of common end points u and v. If each l_i *(i = 1,2, . . . ,k)* is equal to l, we will write $\Theta(l^{[k]})$.

Definition 1.7: A binary tree is defined as a tree in which there is exactly one vertex (root vertex) of degree two and each of the remaining vertices is of degree one or three. A d- level complete binary tree (complete binary tree of depth d) is a binary tree in which each internal vertex (non-pendent vertex) other than the root is of degree three and all the pendent vertices are at level d. (In a binary tree a vertex v is said to be at level *l*, if v is at a distance of *l* from the root).

In [3] and [2] R. Ponraj, J. V. X. Parthipan have proved the following results which we will use in the proof of our theorems.

Theorem 1.1[3]: (3, 1) - kite is not a pair sum graph.

Theorem 1.2[2]: The complete bipartite graphs $K_{1,n}$ and $K_{2,n}$ are pair sum graphs.

2. MAIN RESULTS

Theorem 2.1: The (m, 2)-kite is a pair sum graph for all $m \geq 3$.

Proof: Let V be the vertex set and E be the edge set of (m, 2)-kite. Then |V| = |E| = m + 2. Consider the following two cases.

Case 1: m is even. Let m = 2r.

Let $V = \{u_i, v_i / 1 \leq i \leq r\} \cup \{w_1, w_2\}$ and

$E = \{u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq r-1\} \cup \{u_rv_1, v_ru_1, u_rw_1, w_1w_2\}$.

Define a map $f : V \to \{\pm 1, \pm 2, \ldots, \pm(2r+2)\}$ as follows:

$f(u_i) = 2i$; $1 \leq i \leq r-1$

$f(v_i) = -f(u_i)$; $1 \leq i \leq r-1$

$f(u_r) = 1$, $f(v_r) = -1$, $f(w_1) = -(r+1)$, $f(w_2) = (2r+1)$.

Then f is a pair sum labeling.

Case2: m is odd.

Subcase 2.1: $m \geq 9$

Let m = 2r+1 , $r \geq 4$.

Let $V = \{u_i / 1 \leq i \leq r\} \cup \{v_i / 1 \leq i \leq r+1\} \cup \{w_1, w_2\}$ and

$E = \{u_iu_{i+1} / 1 \leq i \leq r-1\} \cup \{v_iv_{i+1} / 1 \leq i \leq r\} \cup \{u_rv_1, v_{r+1}u_1, v_rw_1, w_1w_2\}$.

Define $f : V \to \{\pm 1, \pm 2, \ldots, \pm(2r+3)\}$ as follows:
$f(u_i) = 2i$; $1 \le i \le r-1$
$f(v_i) = -f(u_i)$; $1 \le i \le r-2$
$f(u_r) = 2r-1$, $f(v_{r-1}) = 2r-3$, $f(v_r) = -(2r-2)$, $f(v_{r+1}) = -(2r-1)$
$f(w_1) = -(2r-5)$, $f(w_2) = 2r+3$.
Then f is a pair sum labeling.

Subcase 2.2: for m = 3 , 5 and 7 , a pair sum labeling of (m,2)-kite is given in Figure 1.

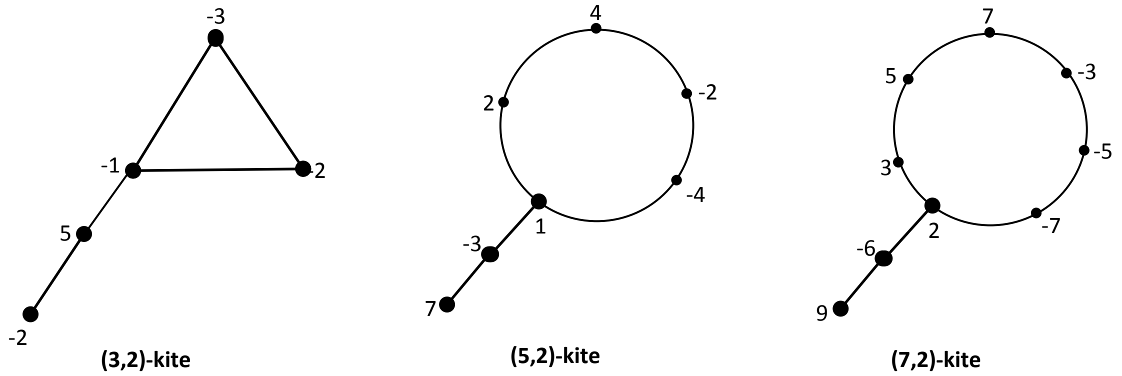

Figure 1

Hence the (m, 2)-kite is a pair sum graph for all $m \ge 3$.

Corollary 2.2: By deleting the pendant vertex in the (m,2)-kite of the proof of the Theorem 2.1 and by Theorem 1.1, the (m, 1)-kite is a pair sum graph except for m = 3.

Example 2.1: In Figure 2, pair sum labeling of (16,2)-kite and (17,2)-kite are illustrated.

Subcase 1.1: m is even. Let m = 2r.Then $V(G) = \{u_i, v_i / 1 \le i \le r\} \cup \{w_i / 1 \le i \le n\}$ and
$E(G) = \{u_r v_1, u_i u_{i+1}, v_i v_{i+1} / 1 \le i \le r-1\} \cup \{w_i / 1 \le i \le n\}$.
Define $f : V \to \{\pm 1, \pm 2, \ldots, \pm(2r+n)\}$ as follows:
$f(u_1) = -3$, $f(u_r) = 1$, $f(v_r) = -1$
$f(u_i) = 2i$; $2 \le i \le r-1$
$f(v_i) = -f(u_i)$; $1 \le i \le r-1$

$$f(w_i) = \begin{cases} 7 & ; \quad i = 1 \\ -(2i+3) & ; \quad 2 \le i \le n \,\&\, i \text{ even} \\ (2i+3) & ; \quad 3 \le i \le n \,\&\, i \text{ odd} \end{cases}$$

Subcase 1.2: m is odd. Let m = 2r+1.
Then $V(G) = \{u_{r+1}, u_i, v_i / 1 \le i \le r\} \cup \{w_i / 1 \le i \le n\}$ and
$E(G) = \{u_i u_{i+1}, u_r v_1, v_r v_{r+1}, v_i v_{i+1} / 1 \le i \le r-1\} \cup \{v_{r+1} w_i / 1 \le i \le n\}$.

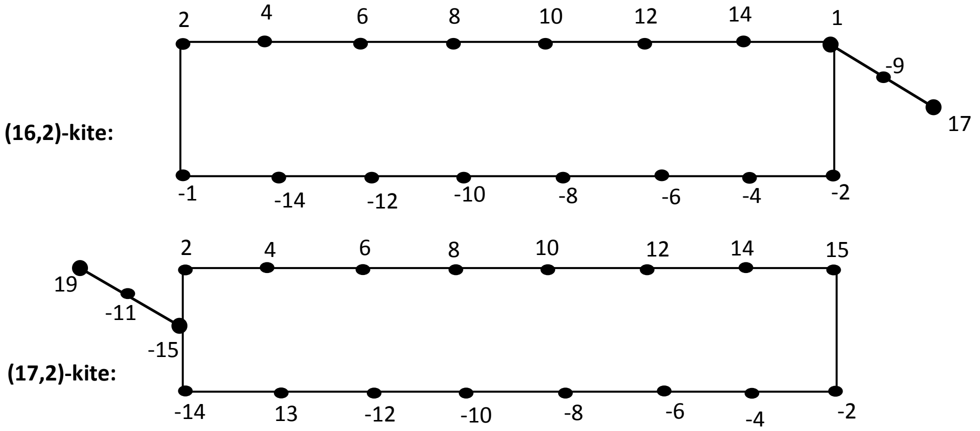

Figure 2: Pair sum labeling of (16,2)-kite and (17,2)-kite

Theorem 2.3: The coconut tree CT(m, n) is a pair sum graph.

Proof: Let G(V,E) = CT(m, n). then $|V(G)| = m + n$ and $|E(G)| = m + n-1$

Case 1: $m \ge 5$

Define $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm(2r+n+1)\}$ as follows:

$$f(u_i) = \begin{cases} 2(i+1) & ; \quad 1 \le i \le r-1 \\ 1 & ; \quad i = r. \end{cases}$$

$$f(v_i) = \begin{cases} -2i & ; \quad 1 \le i \le r \\ -1 & ; \quad i = r+1 \end{cases}$$

$$f(w_i) = \begin{cases} 2 & ; \quad i = 1 \\ 7 & ; \quad i = 2 \\ -(2i+1) & ; \quad 3 \le i \le n \end{cases}$$

Case 2: $m \le 4$.

Subcase 2.1: m =1, then G = CT(1,n) is the star $K_{1,n}$. By Theorem 1.2 [2], G is a pair sum graph.

Subcase 2.2: m =2, then G = CT(2,n) is the star $K_{1,n+1}$. By Theorem 1.2 [2], it is a pair sum graph.

Subcase 2.3: m =3, then G = CT(3,n).

Let $V(G) = \{v_i / 1 \le i \le 3\} \cup \{w_i / 1 \le i \le n\}$ and $E(G) = \{v_1v_2, v_2v_3\} \cup \{v_3w_i / 1 \le i \le n\}$.

Label the vertices v_1, v_2 and v_3 by -3, 2 and -1 respectively.

For $1 \le i \le n$, define $f(w_i) = \begin{cases} -(i+1) & ; \text{ i odd} \\ (i+2) & ; \text{ i even} \end{cases}$

Subcase 2.4: m = 4 then G = CT(4,n).

$V(G) = \{v_i / 1 \le i \le 4\} \cup \{w_i / 1 \le i \le n\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4\} \cup \{v_4w_i / 1 \le i \le n\}$. Label the vertices v_1, v_2, v_3 and v_4 by -2,-4, 1 and 2 respectively.

For $1 \le i \le n$, define

$$f(w_i) = \begin{cases} 4 & ; i = 1 \\ -1 & ; i = 2 \\ -3 & ; i = 3 \\ i-1 & ; 4 \le i \le n \ \& \text{ i even} \\ -(i+2) & ; 5 \le i \le n \ \& \text{ i odd} \end{cases}$$

Then f is a pair sum labeling.

Hence the coconut tree CT(m, n) is a pair sum graph.

Example 2.2: In Figure 3, the pair sum labeling for CT(4,7) is exhibited.

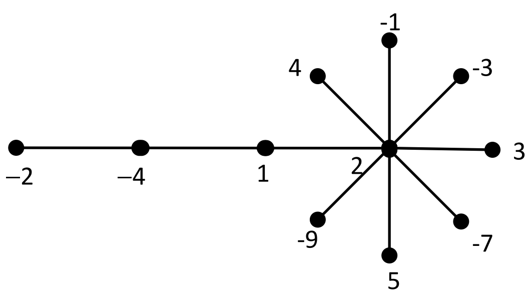

Figure. 3: Pair sum labeling CT(4,7)

Corollary 2. 4: For $r \ge 3$, the Y –tree Y_{r+1} is a pair sum graph.

Proof: It is easily observed that $Y_{r+1} \cong CT(r-1,2)$ for $r \ge 3$. Hence by Theorem 2.1 , Y_{r+1} is a pair sum graph.

Example 2.3: In Figure 4 a pair sum labeling of Y_{4+1} and Y_{5+1} are illustrated.

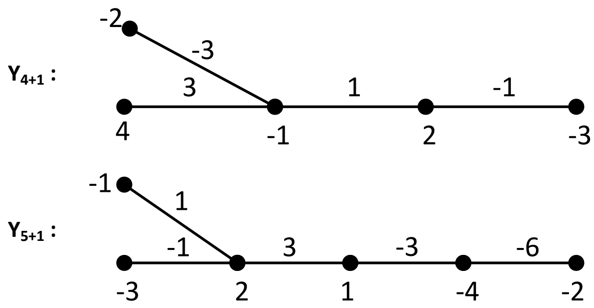

Figure 4: pair sum labeling of Y_{6+1}, Y_{4+1} and Y_{5+1}

Theorem 2.5: For m, n $\ge$1, Jelly fish graph J(m, n) is a pair sum graph.

Proof: Let G(V,E) = J(m, n). Then G has (m+n+4) vertices and (m+n+5) edges. Let $n \ge m$. Let $V(G) = V_1 \cup V_2$ where $V_1 = \{x, u, y, v\}$, $V_2 = \{u_i, v_j ; 1 \le i \le m, 1 \le i \le n\}$ and $E = E_1 \cup E_2$, where $E_1 = \{xu, uy, yv, vx, xy\}$, $E_2 = \{uu_i, vv_j ; 1 \le i \le m, 1 \le j \le n\}$.

Define $f: V \rightarrow \{\pm1, \pm2, \ldots, \pm(m+n+4)\}$ as follows:

Label the vertices x, u, y, v by –5, –1, 3 and 1 respectively.

Define

$$f(v_i) = \begin{cases} 5 & ; \quad i = 1 \\ 2(i-1) & ; \quad 2 \le i \le 4 \\ (i+2) & ; \quad 5 \le i \le m+1 \end{cases}$$

For $2 \le j \le n-m$

$$f(v_{m+j}) = \begin{cases} m+3+(j/2) & ; \quad 2 \le j \le m-n \ \& \text{ j even} \\ -(m+5+(j-1)/2) & ; \quad 3 \le j \le m-n \ \& \text{ j odd} \end{cases}$$

$f(u_i) = -f(v_{i+1})$; $1 \le i \le m$.

Then f is a pair sum labeling. Hence the Jelly fish graph J(m, n) is a pair sum graph.

Example 2.4: In Figure 5, the pair sum labeling for the Jelly fish graph J(4, 7) is exhibited.

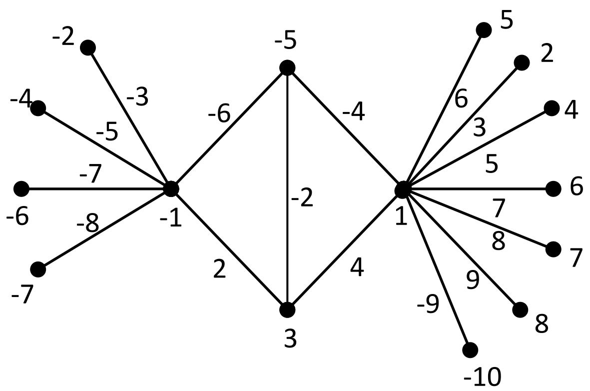

Figure 5: pair sum labeling for the Jelly fish J(4, 7)

Next we will prove that the theta graph $\Theta(\ell^{[m]})$ is a pair sum graph, for m even.

Theorem 2.6: For m even, the theta graph $\Theta(\ell^{[m]})$ is a pair sum graph.

Proof: Let G(V,E) = $\Theta(\ell^{[m]})$. Then $|V(G)| = m(\ell - 1) + 2$ and $|E(G)| = m\ell$. Let $V(G) = V_1 \cup V_2$ where

$V_1 = \{u_{i,j}, v_{i,j} \mid 1 \le i \le m/2 \text{ and } 1 \le j \le \ell - 1\}$,

$V_2 = \{u, v\}$ and $E(G) = E_1 \cup E_2$ where

$E_1 = \{u_{i,j}u_{i,j+1}, v_{i,j}v_{i,j+1} / 1 \le i \le m/2, 1 \le j \le \ell - 2\}$,

$E_2 = \{uu_{i,1}, u\ v_{i,\ell-1}, vv_{i,1}, v\ u_{i,\ell-1} / 1 \le i \le m/2\}$.

Define a map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm m(\ell - 1) + 2\}$ as follows:

$f(u) = 1$, $f(v) = -1$,

$f(u_{i,j}) = 2(\ell i - i + j - \ell + 1)$; $1 \le i \le m/2$ & $1 \le j \le \ell - 1$

$f(v_{i,j}) = f(u_{i,j}))$; $1 \le i \le m/2$ & $1 \le j \le \ell - 1$

Clearly f is a pair sum labeling.Hence the theta graph $\Theta(\ell^{[m]})$ is a pair sum graph for m even.

Example 2.5: In Figure 6, a pair sum labeling for the theta graph $\Theta(7^{[6]})$ is exhibited.

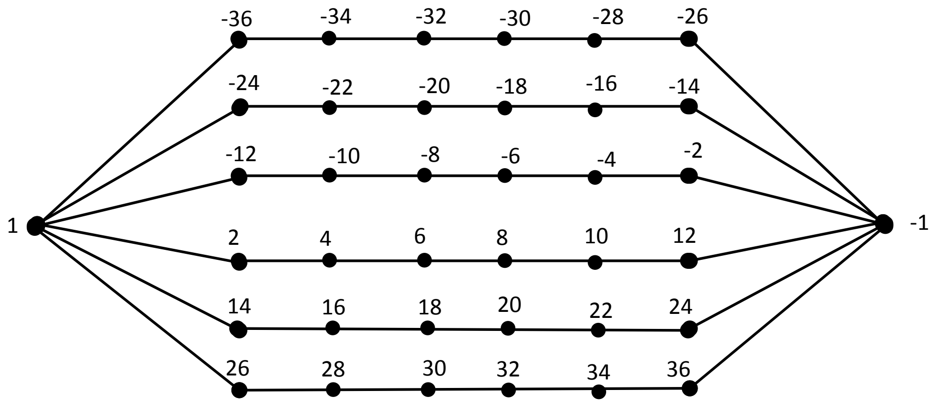

Figure 6: pair sum labeling for the theta graph $\Theta(7^{[6]})$

Next we will prove that a complete binary tree is a pair sum graph. In a binary tree a vertex v is said to be at level d, if v is at a distance of d from the root. The number of vertices at levels 0,1,2,…,m in a m-level complete binary tree are $2^0, 2^1, 2^2, \ldots, 2^m$ respectively. Thus the total number of vertices in m-level complete binary tree is $2^{m+1} - 1$.

Example 2.6: A 3-level complete binary tree is shown in Figure 7.

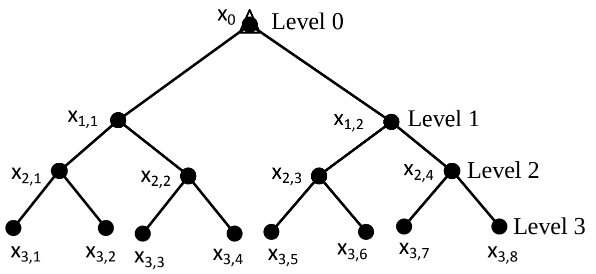

Figure 7 : 3-level complete binary tree

Theorem 2.7: For all $m \ge 1$, the m-level complete binary tree is a pair sum graph.

Proof: Construct the m-level complete binary tree as shown in figure 2.6. Let the vertex set V and the edge set E be defined as $V = \{x_0\} \cup \{x_{i,j} / 1 \le i \le m, 1 \le j \le 2^i\}$ and $E = \{x_0x_{1,1}, x_0x_{1,2}\} \cup \{x_{i,j}x_{i+1,2j-1}, x_{i,j}x_{i+1,2j} / 1 \le i \le m-1, 1 \le j \le 2^i\}$.Then $|V| = 2^{m+1}-1$ and $|E| = 2^{m+1}-2$.

Consider the following two cases.

Case 1: $m \ge 2$.

Define $f : V \to \{\pm 1, \pm 2, \ldots, \pm (2^{m+1}-1\}$ as follows:

$f(x_0) = 1$, $f(x_{1,1}) = 3$, $f(x_{1,2}) = -3$, $f(x_{2,1}) = -2$, $f(x_{2,2}) = -7$, $f(x_{2,3}) = 2$, $f(x_{2,4}) = 5$.

For $1 \le i \le m$, $1 \le j \le 2^i$

Subcase 1.1: i is odd.

$$f(x_{i,j}) = \begin{cases} -2^i - j + 1 & ; \quad 1 \le j \le \frac{1}{2}2^i \\ 2^{i-1} + j - 1 & ; \quad \frac{1}{2}2^i + 1 \le j \le \frac{3}{4}2^i \\ 2^{i-1} + j + 1 & ; \quad \frac{3}{4}2^i + 1 \le j \le 2^i \end{cases}$$

Subcase 1.2: i is even.

$$f(x_{i,j}) = \begin{cases} -2^i - j + 1 & ; \quad 1 \le j \le \frac{1}{4}2^i \\ -(2^i + j + 1) & ; \quad \frac{1}{4}2^i + 1 \le j \le \frac{1}{2}2^i \\ 2^{i-1} + j - 1 & ; \quad \frac{1}{2}2^i + 1 \le j \le 2^i \end{cases}$$

Then it is clearly observed that f is a pair sum labeling.

Case 2: Let m = 1. Then $V = \{x_0, x_{1,1}, x_{1,2}\}$ and $E = \{x_0x_{1,1}, x_0x_{1,2}\}$. Define $f : V \to \{\pm 1, \pm 2, \pm 3\}$ by $f(x_0) = -2$, $f(x_{1,1}) = 3$, $f(x_{1,2}) = 1$. Clearly f is a pair sum labeling.

Hence for all $m \ge 1$, the m-level complete binary tree admits a pair sum labeling.

Example 2.7: In Figure 8, pair sum labeling of four-level complete binary tree is illustrated.

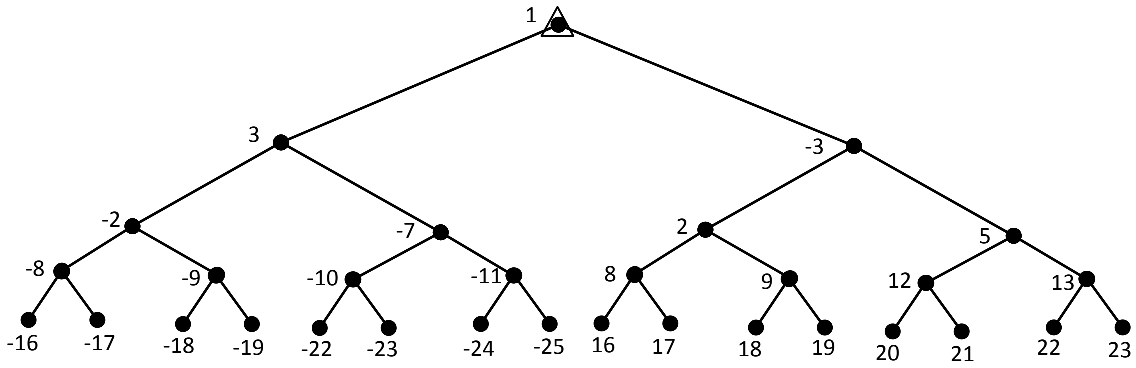

Figure 8: pair sum labeling of four level complete binary tree

3. CONCLUSION

In this paper we investigated some graphs like the coconut tree CT(m, n), the Y-tree Yn+1, Jelly fish graph J(m, n), the (m,2)-kite, (m,1)-kite, the theta graph $\Theta(l[m])$, for m even and complete binary tree are pair sum graphs. For the graphs like trees, (m, n)-kites for all n >2 this labeling can be verified.

4. ACKNOWLEDGEMENTS

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