Common Fixed Point Theorem with Refined Condition of Weak Contraction by Generalized Altering Distance Function

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ABSTRACT
In the present paper, the authors have obtained a unique fixed point theorem for four maps using generalized altering distance function in four variables by considering a refined form of weak contraction than the form used in Theorem 2.1 of [17] which reduces the computational part quite significantly.

AMS Subject Classification: 47H10;60H05;60H10

Keywords
Fixed point, compatible of type (P) or (β) mappings,
Altering distance function, weak contraction.

1. INTRODUCTION
Banach contraction principle has been widely studied by many mathematicians during the last decades (cf. [2], [3], [4], [5], [8], [9], [10], [11], [13], [14] etc.). By considering the weak form of contractive maps, a fixed point problem was initiated by Khan in (cf. [12]). It may be noted that the concept of altering distance function was first introduction by Khan (cf. [12]).

Recently, the concept of altering distance function has become more popular and by using this concept many mathematicians tackled the fixed point problem with the aid of this new concept (cf. [1], [7], [17], [19], [20], [21]). Choudhary (cf. [7]) introduced a generalized distance function in three variables. Rao, K.P.R. et al. in 2007 (cf. [17]) have proved an interesting result by considering generalized altering distance functions of four variables and concluded that the four continuous maps have a unique common fixed point.

In the present paper, the authors have noticed that the condition (i) of Theorem 2.1 of [17] can be replaced by another general condition which significantly reduces the computational part of the proof and established the same conclusion as given in Theorem 2.1 of [17]. In particular, condition (i) of Theorem 2.1 of [17] would be a special case of the main result of this paper. It may be further noted that by choosing suitable values of the variables the result covers several other important theorems also.

2. PRELIMINARIES
In order to establish the main result, the following definitions are required:

Definition 2.1 Let (X, d) be a metric space. A mapping T: X $\longrightarrow$ X is said to be weakly contractive if

$$d(Tx,Ty) \le d(x,y) - \phi d(x,y) \quad \forall \ x,y \in X$$

where $\phi:[0,\infty) \to [0,\infty)$ is an altering distance function.

Definition 2.2 Let S and T be two self maps on a metric space (X, d). The pair (S, T) is said to be **compatible of type (P) or (β)**, if $\lim d(S^2x_n, T^2x_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$. It is clear that STx = TSx, whenever Sx=Tx for any $x \in X$.

Definition 2.3 Let ψ_n denote the set of all functions $\psi:[0,\infty)^n \longrightarrow [0,\infty)$ such that

(i) ψ is continuous

(ii) ψ is monotonic increasing.

(iii) $\psi(t_1, t_2, \ldots, t_n) = 0 \Leftrightarrow t_1 = t_2 = \ldots = t_n = 0$

The functions in ψ_n are called **generalized altering distance functions.**

3. RESULTS ALREADY PROVED

Theorem 3.1 (cf. [7]) Let S and T be self mappings of a complete metric space (X, d) satisfying

$$\phi_1(d(Sx,Ty)) \le \psi_1(d(x,y), d(x,Sx), d(y,Ty)) - \psi_2(d(x,y), d(x,Sx), d(y,Ty))$$

for all x, $y \in X$, where ψ_1 and ψ_2 are generalized altering distance functions and

$$\phi_1(a) = \psi_1(a,a,a), \ \forall a \in [0,\infty).$$

Then S and T have a unique common fixed point in X.

Theorem 3.2 (cf. [17]) Let P, Q, S and T be self mappings of a complete metric space (X, d) satisfying

(i)
$$\phi_1\left(d(Px,Qy)\right) \le \psi_1\left(d(Sx,Ty), d(Sx,Px), d(Ty,Qy), \frac{1}{2}\left[d(Sx,Qy)+d(Ty,Px)\right]\right)$$
$$-\psi_2\left(d(Sx,Ty), d(Sx,Px), d(Ty,Qy), \frac{1}{2}\left[d(Sx,Qy)+d(Ty,Px)\right]\right)$$

for all x, $y \in X$, where $\psi_1, \psi_2 \in \psi_4$ and $\phi_1(a) = \psi_1(a,a,a,a), \ \forall a \in [0,\infty)$.

(ii) Either S and T or P and T or Q and S are continuous.

(iii) (P, S) and (Q, T) are compatible pairs of type (β).

(iv) PT(X) $\bigcup$ QS(X) $\subseteq$ ST(X) and ST = TS.

Then P, Q, S and T have a unique common fixed point in X.

4. MAIN RESULT

Theorem 4.1 Let P, Q, S and T be self-mappings of a complete metric space (X, d) satisfying

(i) $\phi_1(d(Px,Qy)) \le \psi_1(d(Sx,Ty), r_1, r_2, r_3) - r$

for all x, $y \in X$, where $\psi_1 \in \psi_4$, $r \ge 0$, $0 \le r_i \le d(Px,Qy), \ i = 1, 2, 3$.

and $\phi_1(a) = \psi_1(a,a,a,a), \ \forall a \in [0,\infty)$

(ii) Either S and T or P and T or Q and S are continuous.

(iii) (P, S) and (Q, T) are compatible pairs of type (β).

(iv) PT(X) $\bigcup$ QS(X) $\subseteq$ ST(X) and ST = TS.

Then P, Q, S and T have a unique common fixed point in X.

Proof. Let z be any arbitrary point of X. From (iv) construct the sequences $\{x_n\}$ and $\{y_n\}$ of point of X such that

$PTx_{2n} = STx_{2n+1} = y_{2n+1}$, $QSx_{2n+1} = STx_{2n+2} = y_{2n+2}$, $\forall$ n = 0, 1, 2, . . .

Let $a_n = d(y_n, y_{n+1})$. Putting $x = Tx_{2n}$, $y = Sx_{2n+1}$ in (i) leads to

$$\phi_1(a_{2n+1}) \le \psi_1(a_{2n}, r_1, r_2, r_3) - r \tag{4.1}$$

If $a_{2n} < a_{2n+1}$, then

$$\phi_1(a_{2n+1}) \le \psi_1(a_{2n+1}, a_{2n+1}, a_{2n+1}, a_{2n+1}) - r$$
$$< \psi_1(a_{2n+1}, a_{2n+1}, a_{2n+1}, a_{2n+1}) = \phi_1(a_{2n+1})$$

which is a contradiction. Hence $a_{2n+1} \le a_{2n}$, $\forall$ n=0, 1, 2 . . .

Similarly, by putting $x = Tx_{2n}$, $y = Sx_{2n-1}$, in (i) it has been obtained that $a_{2n} \le a_{2n-1}$, $\forall$ n = 1, 2, . . .

Thus, $a_{n+1} \le a_n$, $\forall$ n = 1, 2, . . . , so that $\{a_n\}$is monotonically decreasing sequence of non-negative real numbers and hence converges to a point $a \in R$.

Now, letting $n \to \infty$, (4.1) reduces to

$$\phi_1(a) \le \psi_1(a, 0, 0, 0) - r$$
$$\le \psi_1(a, a, a, a) - r = \phi_1(a), \text{ if } a > 0$$

Thus, $\phi_1(a) < \phi_1(a)$, which is a contradiction, therefore a = 0

Hence, $\lim d(y_n, y_{n+1}) = \lim a_n = 0$ (4.2)

To show that the sequence $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in view of (4.2).

Suppose, if possible, the sequence $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\in > 0$ and monotonically increasing sequences of natural numbers {2m(k)} and {2n(k)} such that n(k) > m(k),

$$d(y_{2m(k)}, y_{2n(k)}) \ge \in \quad \text{and} \quad d(y_{2m(k)}, y_{2n(k)-1}) < \in \tag{4.3}$$

$$\in \le d\left(y_{2m(k)}, y_{2n(k)}\right)$$
$$\le d\left(y_{2m(k)}, y_{2n(k)-1}\right) + d\left(y_{2n(k)-1}, y_{2n(k)}\right)$$
$$< \in + d\left(y_{2n(k)-1}, y_{2n(k)}\right) \quad \text{(using (4.3))}$$

Letting $n \to \infty$ and using (4.2), it may be noted that
$$\lim d\left(y_{2n(k)}, y_{2m(k)}\right) = \in \tag{4.4}$$

Letting $n \to \infty$, using (4.2) and (4.4) in

$$\left| d\left(y_{2n(k)+1}, y_{2m(k)}\right) - d\left(y_{2n(k)}, y_{2m(k)}\right)\right|$$
$$\le d\left(y_{2n(k)}, y_{2n(k)+1}\right)$$

we have, $$\lim d\left(y_{2n(k)+1}, y_{2m(k)}\right) = \in \tag{4.5}$$

Similarly letting $n \to \infty$ and using (4.2) and (4.4) in

$$\left|d\left(y_{2n(k)}, y_{2m(k)-1}\right) - d\left(y_{2n(k)}, y_{2m(k)}\right)\right|$$
$$\le d\left(y_{2m(k)}, y_{2m(k)-1}\right),$$

it has been obtained that,

$$\lim d\left(y_{2n(k)}, y_{2m(k)-1}\right) = \in \tag{4.6}$$

Putting $x = Tx_{2n(k)}$, $y = Sx_{2m(k)-1}$ in (i), leads to

$$\phi_1\left(d\left(y_{2n(k)+1}, y_{2m(k)}\right)\right)$$
$$\le \psi_1\left(d\left(y_{2n(k)}, y_{2m(k)-1}\right), r_1, r_2, r_3\right) - r$$

Letting $k \to \infty$ and using (4.4) and (4.6), it may be noted that

$$\phi_1(\in) \le \psi_1(\in, 0, 0, 0) - r$$
$$< \psi_1(\in, \in, \in, \in) = \phi_1(\in)$$

Which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Let $Tx_{2n} = v_n$, $Sx_{2n+1} = w_{n+1}$, $\forall$ n

Then, $Pv_n \to z$, $Sv_n \to z$, $Qw_{n+1} \to z$ and $Tw_{n+1} \to z$.

Case I. Suppose S and T are continuous.

Step 1. Since S is continuous, $SPv_n \to Sz$, $S^2v_n \to Sz$. Since (P,S) is compatible of type (β), hence $P^2v_n \to Sz$.

Suppose $Sz \neq z$.

Putting $x = Pv_n$, $y = w_{n+1}$ in (i), yields

$$\phi_1\left(d\left(P^2v_n, Qw_{n+1}\right)\right)$$
$$\leq \psi_1\left(d\left(SPv_n,\ Tw_{n+1}\right), r_1, r_2, r_3\right) - r$$

Letting $n \to \infty$, it may be noted

$$\phi_1(d(Sz,z)) \leq \psi_1\left(d(Sz,z), 0, 0, 0\right) - r$$

$$< \psi_1\left(d(Sz,z),\ d(Sz,z), d(Sz,z), d(Sz,z)\right)$$
$$= \phi_1\left(d(Sz,z)\right)$$

It is a contradiction, hence Sz = z.

Putting x = z, $y = w_{n+1}$ in (i) and letting $n \to \infty$ gives, Pz = z.

Step 2. Since T is continuous, we have $TQw_{n+1} \to Tz$, $T^2w_{n+1} \to Tz$. Since (Q,T) is compatible of type (β), leads to have $Q^2w_{n+1} \to Tz$. Putting $x = v_n$, $y = Qw_{n+1}$ in (i) and letting $n \to \infty \Rightarrow$ Tz = z. Again substituting $x = v_n$, y = z in (i) and letting $n \to \infty \Rightarrow$ Qz = z.

Thus, Pz = Qz = Sz = Tz = z.

Case II. Suppose P and T are continuous. From step 2, it may be observed that Tz = Qz = z.

Step 3. Since P is continuous, implies $PSv_n \to Pz$ and $P^2v_n \to Pz$. Since (P,S) is compatible of type (β), we get $S^2v_n \to Pz$. Putting $x = Sv_n$, $y = w_{n+1}$ in (i) and letting $n \to \infty$, yields Pz = z.

Step 4. Now PTz = Pz = z. Since $PT(X) \subseteq ST(X)$, there exists $u \in X$ such that PTz = STu. Let Tu = v so that z = PTz = STu = Sv.

Suppose $Pv \neq z$

Putting x = v, y = z in (i), leads to

$$\phi_1\left(d(Pv,z)\right) = \phi_1\left(d(Pv, Qz)\right)$$
$$\leq \psi_1\left(d(Sv,\ Tz), r_1, r_2, r_3\right) - r$$
$$= \psi_1\left(0,\ r_1, r_2, r_3\right) - r$$
$$\leq \psi_1\left(d(Pv,\ z), d(Pv,\ z), d(Pv,\ z), d(Pv,\ z)\right) - r$$

$$< \psi_1\left(d(Pv,\ z), d(Pv,\ z), d(Pv,\ z), d(Pv,\ z)\right)$$
$$= \phi_1\left(d(Pv,z)\right)$$

It is a contradiction. Hence Pv = z. Since Pv = Sv = z and (P,S) is compatible of type (β), gives Pz = Sz.

Thus, Pz = Qz = Tz = Sz = z.

Case III. Suppose Q and S are continuous. From step 1, it implies Sz = Pz = z. As in step 3, it may be concluded that Qz = z. Since $QS(X) \subseteq ST(X)$, as in step 4, leads to Qz = Tz. Thus, Pz = Qz = Sz = Tz = z.

Uniqueness of the common fixed point follows easily from (i).

5. CONCLUSION

Considering, r_1=d(Sx, Px), r_2=d(Ty, Qy)

$r_3 = \frac{1}{2}\left[d(Sx,Qy) + d(Ty,Px)\right]$ and

r = $\psi_2\left(d(Sx,Ty), d(Sx,Px), d(Ty,Qy),\ \frac{1}{2}\left[d(Sx,Qy) + d(Ty,Px)\right]\right)$

The result of Rao K.P.R. et. al. [17] may be obtained.

Since, the parameters r_1, r_2, r_3 and r are left arbitrary, one may select these according to the demand of applications, for example; in the solution of differential equations.

6. ACKNOWLEDGMENTS

The authors would like to thank referees of this paper for their helpful suggestions.

The second author would like to thank University Grants Commission for supporting this work under the teacher fellowship program.

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