Chaos Suppression in Forced Van Der Pol Oscillator

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ABSTRACT
This paper presents a new method of controlling chaos in the nonlinear Van Der Pol oscillator with uncertainties. The proposed method is based on a nonlinear observer to estimate unmeasured velocity signal coupled to a control law. The observer ensures, firstly, an asymptotic convergence of the velocity estimation error. Then, the control law, which is based on the estimated variables, forces the output system to track a desired trajectory despite presence of uncertainties (external forces) on the system dynamics. Simulation results are provided to show the effectiveness of the proposed control strategy.

General Terms
Control theory, mathematic modeling, signal processing.

Keywords
Control, observer design, chaotic oscillator, uncertainties.

1. INTRODUCTION
In the few last decades, controlling chaos has received a great interest [1-3]. In many applications, chaos has been viewed as an undesirable phenomenon which may damage such physical systems, especially in mechanical non linear devices such as coupled oscillators [4]. The first control strategy was suggested by Ott et al. [1] in order to stabilize the unstable periodic orbits. After then, different methods has been developed for controlling chaotic systems [5-7] and [8]. Zeng et al. [9] proposed an adaptive controller to control chaos in Lorenz System. In [10], Kotaro et al. developed a neural networks based control law for chaotic systems. However, many of these proposed methods supposed knowledge of the all state variables which can not be always measured due to noise that affect sensors. Consequently, the design of a state-observer is needed to estimate the unmeasured velocity signals of such a system in order to construct the adequate control law. In literature, several types of observers have been proposed for chaotic systems [11-13]. In [14], the authors proposed an observer-based Backstepping control scheme to stabilize a class of chaotic systems. These approaches seem to give good results on controlling chaos, however, many of them fail for dynamical systems in presence of external forces (perturbing terms). In this paper, we propose a novel observer based control scheme to suppress chaos in forced Van Der Pol oscillator. The control strategy is based on a novel sliding mode observer to estimate the unmeasured velocity signal of the system. This observer is, then, coupled to a control law based on flatness of the system dynamics and which forces the output system to track a desired trajectory. The global tracking problem is, finally, solved despite the presence of perturbing external forces in the oscillator dynamics. The reminder of this paper is organized as follows. Section II displays the mathematical model of the Van Der Pol oscillator and underlines its chaotic behaviour. Section III is devoted to the development of the flatness based control law. In section IV, we present the observer design and the asymptotic convergence analysis. Section V illustrates the main results when applying our proposed method to stabilize the unstable periodic orbits of the chaotic Van Der Pol oscillator. Finally, some conclusions are included in Section VI.

2. PROBLEM STATEMENT
The dynamics of the system under consideration belongs to the class of the uncertain chaotic system described by the following equation:

$$\ddot{x} = f(x,\dot{x},t) + f_e(t) + u(t) \quad (1)$$

where x and $\dot{x}$ represent respectively the position and its *i*th derivatives of the oscillator. $f(x,\dot{x},t)$ is an unknown nonlinear function, $f_e(t)$ is an unknown external perturbing term and *u(t)* is the control input to be determined. This class of systems includes a wide variety of chaotic oscillators which may present the coexistence of chaotic attractors. The mathematical model of the Van Der Pol oscillator is given by:

$$\ddot{x} + \mu(x^2-1).\dot{x} + x = q.\cos(w.t) + u(t) \quad (2)$$

where μ, q and *w* are nonzero constant parameters. Different works [5] have shown that for various values of these parameters, the forced Van der Pol oscillator may exhibit a wide variety of nonlinear behaviour, including chaos. Let us choose $x_1 = x, x_2 = \dot{x}$. Then, model (2) can be rewritten in the following state-representation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\mu(x_1{}^2-1).x_2 - x_1 + q.\cos(w.t) + u(t) \end{cases} \quad (3)$$

For the following parameters $\mu = 6, q = 2.5$, and $w = 3$, it was shown in [5] that the behavior of the Van der Pol oscillator is chaotic in the absence of control law as shown in

Figure 1, from which, we can see that the states of the system are always bounded inside the region $x_1 \in (-2,2)$ and $x_2 \in (-5,5)$.

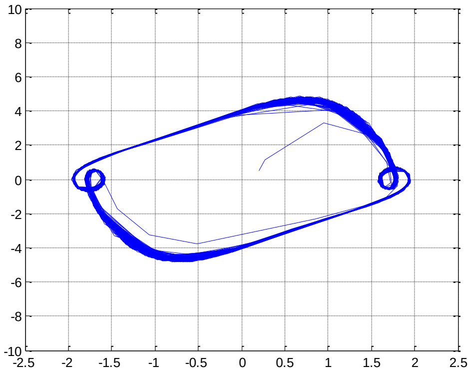

Figure 1. State trajectory of Van der Pol oscillator before control (strange attractor)

In order to avoid fracture of the mechanical parts and some undesirable dynamical effects, it is recommended to induce regular dynamics in this class of systems. Thus, it is necessary to introduce a control action in the system dynamics. However, in practice, velocity sensors are always contaminated by noise due to operational or environmental conditions. It is therefore necessary to use a state-observer in order to estimate the system variables, from only position measurements, and then construct the control law.

3. CONTROL LAW

In this section, we will use the concept of flatness of the chaotic oscillator to construct the control law. So, we introduce, first, the following definition of flatness

Definition1: The nonlinear system

$$\dot{x} = f(x,u) \qquad (4)$$

where $x \in R^n, u \in R^m$ and f a nonlinear function is said to be flat system [15] if there exists a differentially output function $y = (y_1,...,y_m)$ such that all system variables and state are parameterised in terms of y and a finite number of its time derivatives. y is called flat output of system (4).

Proposition1:
The nonlinear system (2) is a differentially flat system with respect to the flat output $y = x_1$.

Proof

Choosing $y = x_1$ as a flat output and applying definition 1 to the system (2), we get

$$\begin{cases} x_1 = y \\ x_2 = \dot{x}_1 = \dot{y} = \dfrac{dy}{dt} \\ u = \ddot{y} + \mu(y^2 - 1)\dot{y} + y - q\cos(wt) \end{cases} \qquad (5)$$

From (3), we have

$$\begin{aligned} \dot{x}_2 &= -\mu({x_1}^2 - 1).x_2 - x_1 + q\cos(w.t) + u(t) \\ &= -\mu(y^2 - 1).\dot{y} - y + q.\cos(w.t) + u(t) \end{aligned} \qquad (6)$$

From (5), and using the expression of u, we have:

$$\dot{x}_2 = \ddot{y} \qquad (7)$$

Consequently, from (5), (6), and (7), it is clearly seen that the system of variables and states are parameterised in terms of the flat output y and a finite number of its time derivatives.

Now, the control law is based on the flatness of the system (2) and developed using Pole Placement Approach for Tracking.

For the flat output system y, let us consider a given reference trajectory y^*.

Proposition 2.

Let the set of real coefficients k_1, k_2 be chosen so that the polynomial $P(s) = s^2 + k_1 s + k_0$ is Hurwitz. Then, the controller

$$v = v^* + k_1\dot{e} + k_0.e \qquad (8)$$

globally exponentially asymptotically stabilizes the tracking error defined by $e = y - v^*$ where $v^* = \dfrac{d}{dt^2}(y^*)$.

Proof:

From (7), we demonstrate that there exist a diffeomorphism such that $\ddot{y} = v$. Then, the error tracking system is given by

$$\ddot{e}(t) = \ddot{y}(t) - \ddot{y}^*(t) = v - v^* \qquad (9)$$

Using (8), we have

$$\ddot{e}(t) = -k_1\dot{e}(t) - k_0.e(t) \qquad (10)$$

From (10) and using suitable choice of k_1, k_2, it is clearly seen that e(t) converges globally exponentially to zero.

4. DEVELOPMENT OF THE NONLINEAR OBSERVER

The controller (8) is based on the knowledge of the velocity signals and the output system. However, from practical point of view, this is not always releasable and the velocity signals are always contaminated with noise. Then, it is necessary to use a state-observer to estimate the unmeasured velocities. In this section we will develop a novel observer based on the sliding mode technique. To this end, let $\hat{x}_1(t),\hat{x}_2(t)\in\Re^2$ denote the estimated position and velocity of system (2), and the estimation errors $e_1(t),\dot{e}_1(t)\in\Re^2$ be defined, respectively, by

$$e_1=\hat{x}_1-x_1, \tag{11}$$

$$\dot{e}_1=\dot{\hat{x}}_1-\dot{x}_1 \tag{12}$$

Let the signal $r(t)\in\Re$ be the sliding surface defined as

$$r(t)=\alpha.e_1(t)+\dot{e}_1(t) \tag{13}$$

where α is a positive scalar to be chosen, under assumption 1, so that $\mathrm{sgn}(r(t))=\mathrm{sgn}(e_1(t)),\forall t\geq 0$, where sgn(.) is the standard signum function.

Assumption1. The initial conditions of the state vector of the system (2) $[q^T(t_0)\quad \dot{q}^T(t_0)]^T$ and the control force u(t) are chosen so that the position and the velocity vector are bounded functions of time.

By this assumption, the scalar α can be chosen such that $\alpha>\frac{\rho_2}{\rho_1}$ where ρ_1,ρ_2 are two positive constants ρ_1,ρ_2 given by $\|e_1\|_{-\infty}=\rho_1$ and $\|\dot{e}_1\|_{+\infty}=\rho_2$.

After an appropriate choice of the scalar α, we can guarantee that $\mathrm{sgn}(r(t))=\mathrm{sgn}(e_1(t)),\forall t\geq 0$.

In this section, the following assumption is required for our analysis.

Assumption2. The term representing uncertainties $f_e(t)=q\cos(w.t)$ is bounded by a positive constant δ.

Our objective is to ensure an asymptotic convergence of $e_1(t)$ and $\dot{e}_1(t)$ to zero as $t\to\infty$. By assuming that he position x is the only variable system available for measurements, we propose the following dynamic observer for the estimation of the velocity signal of system (2)

$$\begin{cases}\dot{\hat{x}}_1=\hat{x}_2-(\lambda_1+\alpha).e_1\\ \dot{\hat{x}}_2=-\mu(x_1{}^2-1).\hat{x}_2-x_1-\lambda_2.e_1\\ \qquad -\lambda_3.sign(e_1)+u(t)\end{cases} \tag{14}$$

where $\lambda_1,\lambda_2,\lambda_3\in\Re$ are the positive observer gains to be given by theorem 1 as following

$$\begin{cases}\lambda_1>18\\ \lambda_2>\left(\lambda_1(\alpha+18)+1\right)\\ \lambda_3>\delta\end{cases} \tag{15}$$

Proof:

The proof of convergence of the observer (14) to the real system (2) is demonstrated in our work given in [16] and reported here. To demonstrate the asymptotically convergence of the estimation error dynamics to zero, we define, first, a definite positive Lyapunov function V(t). The proposed function is given by

$$V=\frac{1}{2}r^T.r+\frac{1}{2}e_1{}^T.e_1 \tag{16}$$

The objective is to find sufficient conditions on $\lambda_1,\lambda_2,\lambda_3$ so that the time derivative of V is negative definite which make the Lyapunov function continually decreasing. The time derivate of (16) gives

$$\dot{V}=r^T.\dot{r}+e_1{}^T.\dot{e}_1=r^T.(\alpha.\dot{e}_1+\ddot{e}_1)+e_1{}^T.\dot{e}_1 \tag{17}$$

The second derivative of the output error $e_1(t)$ leads to

$$\ddot{e}_1=\ddot{\hat{x}}_1-\ddot{x}_1=\dot{\hat{x}}_2-\dot{x}_2-(\lambda_1+\alpha).\dot{e}_1 \tag{18}$$

Let $e_2=\hat{x}_2-x_2$ be the velocity error of the oscillator. So, using (3) and (14), equation (18) can be rewritten as

$$\begin{aligned}\ddot{e}_1=&-\mu(x_1{}^2-1)e_2-q\cos(wt)\\ &-\lambda_2 e_1-\lambda_3\,\mathrm{sgn}(e_1)-(\lambda_1+\alpha)\dot{e}_1\end{aligned} \tag{19}$$

When replacing (19) into (17), we get

$$\begin{aligned}\dot{V}=&-r^T\Big[\mu(x_1{}^2-1)e_2-q\cos(wt)\\ &-\lambda_2 e_1-\lambda_3\,\mathrm{sgn}(e_1)-\lambda_1\dot{e}_1\Big]+e_1{}^T.\dot{e}_1\end{aligned} \tag{20}$$

Or, from (12), and (14), we have

$$e_2 = \dot{e}_1 + (\alpha + \lambda_1)e_1 \qquad (21)$$

and from (13), we have

$$\dot{e}_1 = r - \alpha e_1 \qquad (22)$$

Using (21) and (22), the system (20) can be rewritten as

$$\begin{aligned}\dot{V} = & -r^T\mu(x_1{}^2 - 1)r - r^T\lambda_1 r \\ & -\lambda_1\mu(x_1{}^2 - 1)r^T e_1 + \alpha\lambda_1 r^T e_1 + e_1^T r - \lambda_2 r^T e_1 \\ & - r^T q\cos(wt) - \lambda_3 r^T \operatorname{sgn}(e_1) - \alpha e_1^T e_1\end{aligned} \qquad (23)$$

Using assumption 1, we have shown that a suitable choice of the scalar α gives $\operatorname{sgn}(r(t)) = \operatorname{sgn}(e_1(t)), \forall t \ge 0$. So, using this property, the time derivate of the Lyapunov function leads, finally, to

$$\begin{aligned}\dot{V} = & -r^T\mu(x_1{}^2 - 1)r - r^T\lambda_1 r \\ & -\lambda_1\mu(x_1{}^2 - 1)r^T e_1 + \alpha\lambda_1 r^T e_1 + e_1^T r - \lambda_2 r^T e_1 \\ & - r^T q\cos(wt) - \lambda_3 r^T \operatorname{sgn}(r) - \alpha e_1^T e_1\end{aligned} \qquad (24)$$

The system parameters were taken as $\mu = 6, q = 2.5$, and $w = 3$. Besides, figure shows that, for these parameters, the system state x_1 is always bounded inside the region $x_1 \in (-2, 2)$. Then, the term $\mu(x_1{}^2 - 1)$ of equation (24) can be bounded by a positive constant equal to 18. Now, under assumption 1 and 2, and using the propriety $x.sign(x) = |x|$, we can upper bound the right-hand side of (24) as follows:

$$\begin{aligned}\dot{V} \le & -\|r\|^2.[\lambda_1 - 18] \\ & -\|r\|.\|e_1\|.\left[\lambda_2 - \left(\lambda_1(\alpha + 18) + 1\right)\right] \\ & -\|r\|.[\lambda_3 - \delta] - \alpha.\|e_1\|^2\end{aligned} \qquad (25)$$

From equation (25), we can clearly seen that if the following conditions are satisfied

$$\begin{cases}\lambda_1 > 18 \\ \lambda_2 > \left(\lambda_1(\alpha + 18) + 1\right) \\ \lambda_3 > \delta\end{cases} \qquad (26)$$

then, we can easily obtain a negative semi definite function in a neighbourhood of the sliding surface defined by $e_1 = \dot{e}_1 = 0$ (i.e. in a neighbourhood of $e_2 = 0$). By Lasalle Theorem, we can guarantee an attractive and invariant sliding surface: the global asymptotic velocity observation is then guaranteed.

So, under conditions of system (26), V(t) is a positive-definite Lyapunov function whose time derivative $\dot{V}(t)$ is negative definite. Now, the main result of this note is given in theorem1.

Theorem1. Provided the conditions of system (26), and under assumptions 1 and 2, the observer (14) ensures a finite time global asymptotically convergence of estimated states to real states of the chaotic oscillator given by system (2), i.e. $(\hat{x}_1, \hat{x}_2) \to (x_1, x_2)$ in finite time.

5. SIMULATION RESULTS

For simulation results, the parameters μ, q and *w* are chosen as $\mu = 6, q = 2.5$, and $w = 3$. For these parameters, it was shown that the behaviour of the Van der Pol oscillator is chaotic in the absence of control as shown in Figure 1. Under theorem 1, the proposed observer gains are chosen as $\lambda_1 = 18$, $\lambda_2 = 2160, \lambda_3 = 3, \alpha = 100$. Gains k_1 and k_2 are chosen to be equal to 10 and 15 respectively. The control objective is to drive the output system (2) to the desired trajectory chosen as $y^* = \sin(t)$. Simulations results given by figures 2, 3 4, and 5 show the efficiency of the proposed method using flatness based control law coupled to the sliding mode observer (14). In fact, we can see, firstly, that the observer provides, for system (2), a good estimation of the velocity signal as shown in figure 2. The finite time convergence of the velocity observation error is then guaranteed. Secondly, when applying the control law based on the concept of flatness and the estimated variables, we can show that the system (2) exhibits the behaviour of limit cycle (stable periodic orbits) as shown in figure 3. Finally, it is clearly seen, from figures 4 and 5, that the controller given by (6) forces the system to track the desired trajectory in a finite time despite existence of perturbing terms. Figure 4 displays the finite time convergence of the tracking position and velocity errors to zero. In figure 5, are shown both the output system and the desired trajectory.

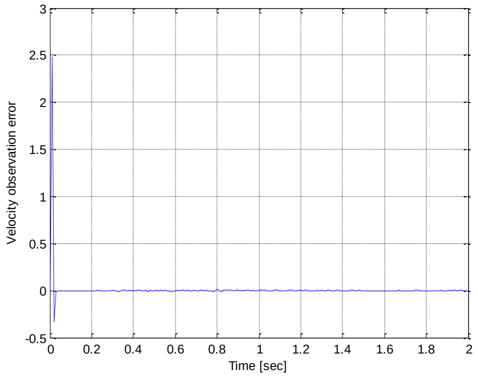

Figure 2 . Velocity estimation error of Van der Pol oscillator

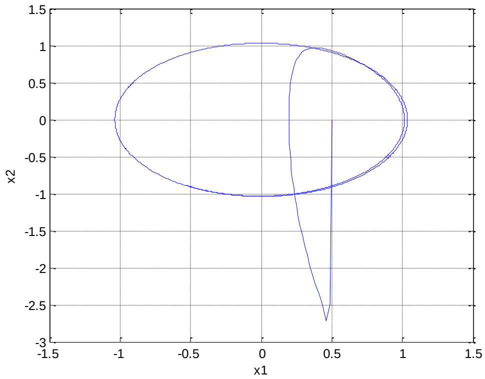

Figure 3. State trajectory of Van der Pol oscillator after control (limit cycle)

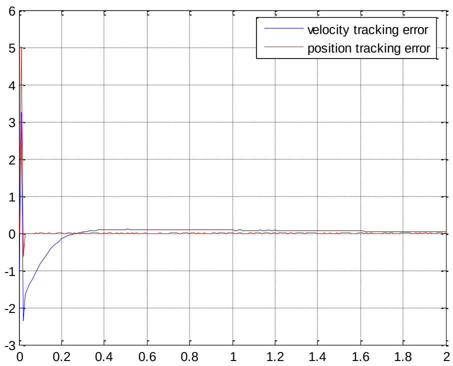

Figure 4. Tracking errors of Van der Pol oscillator: $\hat{x}_1 - y^*$ **and** $\hat{x}_2 - \dot{y}^*$

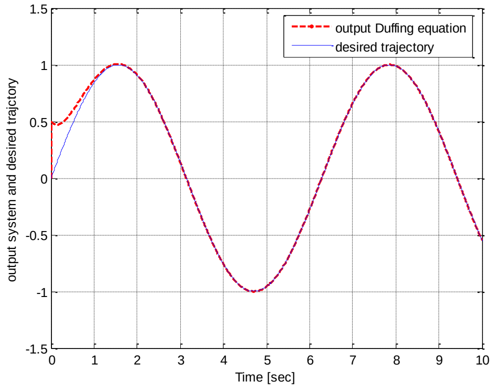

Figure 5. Output system and desired trajectory.

6. CONCLUSIONS

An observer based control law has been proposed, in this paper, to suppress (control) chaos in the forced Van Der Pol oscillator dynamics. The control scheme has been designed by coupling a novel sliding mode observer with a control law using the property of flatness of the system. The proposed method has shown excellent results. Firstly, it was demonstrated through simulations that our proposed approach provides a finite time estimation of the unmeasured velocity signal. Secondly, it has been shown that, when applying our control law, the system output tracks the desired trajectory and the oscillator exhibits the behaviour of stable periodic orbits despite presence of perturbations on the system dynamics. Further works will be done on suppression chaos in switching chaotic systems.

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