Approximate Solution of Wave Equation using Fuzzy Number

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ABSTRACT

In this paper, the fuzzy solution of the initial boundary value problem of hyperbolic one-dimensional wave equation is considered. The solution by finite difference method is observed by using fuzzy intervals.

Keywords:

α-cut, fuzzy membership function, triangular fuzzy number, fuzzy interval

1. INTRODUCTION

In the context of fuzzy theory, first introduced by Zadeh [1], the arithmetic operations on fuzzy numbers are usually approached either by the use of the extension principle (in the domain of the membership function) or by the interval arithmetics (in the domain of the α-cuts). The exact analytical fuzzy operations dates back from the early 1980s and are outlined by Dubois and Prade [2] ; the same authors have introduced the well-known L-R model and the corresponding formulas for the fuzzy operations [3]. The fuzzy differential equations and fuzzy initial value problem were regularly treated by Kaleva [4, 5]. The numerical methods for solving fuzzy differential equations are introduced in [6, 7, 8, 9]. In this paper, first finite difference method to solve one-dimensional wave equation is applied using fuzzy interval arithmetics [10, 11, 12] , numerically.

2. BASIC CONCEPT OF FUZZY SET THEORY

A triangular fuzzy number u can be defined as a triplet $[a, b, c]$. Its membership function is defined as:

$$\mu_u(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \end{cases}$$

To find the α-cut of u, $\alpha\epsilon[0, 1]$ to both left and right reference functions of u is set. That is, $\alpha = \frac{x-a}{b-a}$ and $\alpha = \frac{c-x}{c-b}$. Expressing x in terms of α , $x = (b - a)\alpha + a$ and $x = c - (c - b)\alpha$ is obtained, which gives the α-cut of u is $u_\alpha = [(b - a)\alpha + a, c - (c - b)\alpha]$.
If $u = [u^-, u^+]$ and $v = [v^-, v^+]$ are two given fuzzy numbers, the arithmetic operations are defined for $0 \leq \alpha \leq 1$ as follows:
1. $u = v$ if and only if $u^-_\alpha = v^-_\alpha$ and $u^+_\alpha = v^+_\alpha$.
2. $u + v = [u^-_\alpha + v^-_\alpha, u^+_\alpha + v^+_\alpha]$.
3. $ku = [min\{ku^-_\alpha, ku^+_\alpha\}, max\{ku^-_\alpha, ku^+_\alpha\}]$, $(k\epsilon\Re)$
4. $u + v = [u^-_\alpha - v^+_\alpha, u^+_\alpha - v^-_\alpha]$.

3. APPLICATION OF FINITE DIFFERENCE METHOD IN WAVE EQUATION

Consider the initial boundary value problem of hyperbolic one-dimensional wave equation [13]

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, 0 \leq x \leq L, t \geq 0 \tag{1}$$

initial condition

$$u(x, 0) = f(x) \tag{2}$$

subject to boundary conditions

$$u(0, t) = u(L, t) = 0 \tag{3}$$

with initial velocity

$$u_t(x, 0) = f(x). \tag{4}$$

Using finite difference method and substituting
$u_{tt} = \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})$ and
$u_{xx} = \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$
we get

$$u_{i,j+1} = ru_{i+1,j} + (2 - 2r)u_{i,j} + ru_{i-1,j} - u_{i,j-1} \tag{5}$$

where $r = \frac{a^2k^2}{h^2}$.This explicit method converges and stable for $0 < r \leq 1$. For a given a^2 , choose h and k such that $r = 1$. Hence, (5) becomes

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}. \tag{6}$$

For the initial condition $u_t(x, 0) = f(x)$, central difference approximation for the derivative is used and $u_{i1} = \frac{f_{i+1}+f_{i-1}}{2}$ is written.

4. FUZZIFICATION OF WAVE EQUATION

Let $u_{i-1,j-1} = [W_{01}, W_{02}, W_{03}]$ and $u_{i+1,j-1} = [W_{21}, W_{22}, W_{23}]$.
$[u_{i,j}] = [\frac{W_{01}+W_{21}}{2}, \frac{W_{02}+W_{22}}{2}, \frac{W_{03}+W_{23}}{2}] = [E_{11}, E_{12}, E_{13}]$ is obtained. Firstly, fuzzy membership function for $u_{i-1,j-1}$ is

$$\mu_{u_{i-1,j-1}}(X) = \begin{cases} \frac{X-W_{01}}{W_{02}-W_{01}}, & W_{01} \leq X \leq W_{02} \\ \frac{W_{03}-X}{W_{03}-W_{02}}, & W_{02} \leq X \leq W_{03} \end{cases} \tag{7}$$

Then, $[u_{i-1,j-1}]_\alpha = [W_{01} + (W_{02} - W_{01})\alpha, W_{03} - (W_{03} - W_{02})\alpha]$ is the α-cut of fuzzy number $u_{i-1,j-1}$.

Secondly, fuzzy membership function for $u_{i+1,j-1}$ is

$$\mu_{u_{i+1,j-1}}(X)=\begin{cases}\frac{X-W_{21}}{W_{22}-W_{21}}, & W_{21}\le X\le W_{22}\\ \frac{W_{23}-X}{W_{23}-W_{22}}, & W_{22}\le X\le W_{23}\end{cases} \quad (8)$$

Then, $[u_{i+1,j-1}]_\alpha=[W_{21}+(W_{22}-W_{21})\alpha, W_{23}-(W_{23}-W_{22})\alpha]$ is the α-cut of fuzzy number $u_{i+1,j-1}$.
Hence α-cut for $u_{i,j}$ is [14]

$$[u_{i,j}]_\alpha=\frac{1}{2}\begin{matrix}[\{(W_{01}+W_{21})+\{(W_{02}-W_{01})+(W_{22}-W_{21})\}\alpha,\\ (W_{03}+W_{23})-\{(W_{03}-W_{02})+(W_{23}-W_{22})\}\alpha]\end{matrix}.$$

Let
$X_1=\frac{\{(W_{01}+W_{21})+\{(W_{02}-W_{01})+(W_{22}-W_{21})\}\alpha}{2}$
$\Rightarrow \alpha=\frac{2X_1-(W_{01}+W_{21})}{(W_{02}-W_{01})+(W_{22}-W_{21})}$
and
$X_2=\frac{\{(W_{03}+W_{23})-\{(W_{03}-W_{02})+(W_{23}-W_{22})\}\alpha}{2}$
$\Rightarrow \alpha=\frac{2X_2-(W_{03}+W_{23})}{(W_{03}-W_{02})+(W_{23}-W_{22})}$.
Thus, fuzzy membership function for $u_{i,j}$ is $\mu_{u_{i,j}}(X)=$

$$\begin{cases}\frac{2X-(W_{01}+W_{21})}{(W_{02}+W_{22})-(W_{01}+W_{21})}, & \frac{(W_{01}+W_{21})}{2}\le X\le\frac{(W_{02}+W_{22})}{2}\\ \frac{-2X+(W_{03}+W_{23})}{(W_{03}+W_{23})-(W_{02}+W_{22})}, & \frac{(W_{02}+W_{22})}{2}\le X\le\frac{(W_{03}+W_{23})}{2}\end{cases}$$

Now,let $u_{i-1,j}=[E_{01},E_{02},E_{03}]$, $u_{i+1,j}=[E_{21},E_{22},E_{23}]$ and $u_{i,j-1}=[W_{11},W_{12},W_{13}]$. We get
$[u_{i,j+1}]=[E_{01}+E_{21}-W_{11},E_{02}+E_{22}-W_{12},E_{03}+E_{23}-W_{13}]$. Firstly, fuzzy membership function for $u_{i-1,j}$ is

$$\mu_{u_{i-1,j}}(X)=\begin{cases}\frac{X-E_{01}}{E_{02}-E_{01}}, & E_{01}\le X\le E_{02}\\ \frac{E_{03}-X}{E_{03}-E_{02}}, & E_{02}\le X\le E_{03}\end{cases} \quad (9)$$

and $[u_{i-1,j}]_\alpha=[E_{01}+(E_{02}-E_{01})\alpha, E_{03}-(E_{03}-E_{02})\alpha]$ is the α-cut of fuzzy number $u_{i-1,j}$.
Then, fuzzy membership function for $u_{i+1,j}$ is

$$\mu_{u_{i+1,j}}(X)=\begin{cases}\frac{X-E_{21}}{E_{22}-E_{21}}, & E_{21}\le X\le E_{22}\\ \frac{E_{23}-X}{E_{23}-E_{22}}, & E_{22}\le X\le E_{23}\end{cases} \quad (10)$$

and $[u_{i+1,j}]_\alpha=[E_{21}+(E_{22}-E_{21})\alpha, E_{23}-(E_{23}-E_{22})\alpha]$ is the α-cut of fuzzy number $u_{i-1,j}$.
Then, fuzzy membership function for $u_{i,j-1}$ is

$$\mu_{u_{i,j-1}}(X)=\begin{cases}\frac{X-W_{11}}{W_{12}-W_{11}}, & W_{11}\le X\le W_{12}\\ \frac{W_{13}-X}{W_{13}-W_{12}}, & W_{12}\le X\le W_{13}\end{cases} \quad (11)$$

and $[u_{i,j-1}]_\alpha=[W_{11}+(W_{12}-W_{11})\alpha, W_{13}-(W_{13}-W_{12})\alpha]$ is the α-cut of fuzzy number $u_{i,j-1}$.
Hence,α-cut for $u_{i,j+1}$ is $[u_{i,j+1}]_\alpha=$
$[(E_{01}+E_{21}-W_{13})+\{(E_{02}-E_{01})+(E_{22}-E_{21})+(W_{13}-W_{12})\}\alpha,(E_{03}+E_{23}-W_{11})-\{(E_{03}-E_{02})+(E_{23}-E_{22})+(W_{12}-W_{11})\}\alpha]$.
Let
$X_1=$
$(E_{01}+E_{21}-W_{13})+\{(E_{02}+E_{22}-W_{12})-(E_{01}+E_{21}-W_{13})\}\alpha$
$\Rightarrow \alpha=\frac{X_1-(E_{01}+E_{21}-W_{13})}{(E_{02}+E_{22}-W_{12})-(E_{01}+E_{21}-W_{13})}$
and
$X_2=$
$(E_{03}+E_{23}-W_{11})-\{(E_{03}+E_{23}-W_{11})-(E_{02}+E_{22}-W_{11})\}\alpha$
$\Rightarrow \alpha=\frac{-X_2+(E_{03}+E_{23}-W_{11})}{(E_{03}+E_{23}-W_{11})-(E_{02}+E_{22}-W_{11})}$.
Then, fuzzy membership function for $u_{i,j+1}$ is on the way
$A=E_{01}+E_{21}-W_{13}, B=E_{02}+E_{22}-W_{12}$ and
$C=E_{03}+E_{23}-W_{11}$

$$\mu_{u_{i,j+1}}(X)=\begin{cases}\frac{X-A}{B-A}, & A\le X\le B\\ \frac{C-X}{C-B}, & B\le X\le C\end{cases} \quad (12)$$

With the help of this fuzzy membership function of different u's of $j+1$th level can be found out.

5. NUMERICAL EXAMPLES

Consider the one-dimensional wave equation
$\frac{\partial^2 u}{\partial t^2}=16\frac{\partial^2 u}{\partial x^2}$
with the conditions
$u(x,0)=x^2(5-x)$
$u(0,t)=u(5,t)=0$
$u_t(x,0)=0$
Let $u_{00}=[-0.001,0,0.001]$ and $u_{20}=[11.999,12,12.001]$.
and $u_{11}=[5.999,6,6.001]$.
Firstly, fuzzy membership function for u_{00} is

$$\mu_{u00}(x)=\begin{cases}\frac{x+0.001}{0.001}, & -0.001\le x\le 0\\ \frac{12.001-x}{0.001}, & 0\le x\le 0.001\end{cases}$$

and α-cut for u_{00} is $[u_{00}]_\alpha=[0.001\alpha-0.001, 0.001-0.001\alpha]$
Secondly, fuzzy membership function for u_{20} is

$$\mu_{u20}(x)=\begin{cases}\frac{x-11.999}{0.001}, & 11.999\le x\le 12\\ \frac{12.001-x}{0.001}, & 12\le x\le 12.001\end{cases}$$

and α-cut for u_{20} is $[u_{20}]_\alpha=[0.001\alpha+11.999, 12.001-0.001\alpha]$. Thus, α-cut for u_{11} is $[u_{11}]_\alpha=[0.001\alpha+5.999, 6.001-0.001\alpha]$. Let $X_1=5.999+0.001\alpha\Rightarrow\alpha=\frac{X_1-5.999}{0.001}$ and
$X_2=6.001-0.001\alpha\Rightarrow\alpha=\frac{-X_2+6.001}{0.001}$.
Hence, fuzzy membership function for u_{11} is

$$\mu_{u11}(x)=\begin{cases}\frac{x-5.999}{0.001}, & 5.999\le x\le 6\\ \frac{6.001-x}{0.001}, & 6\le x\le 6.001\end{cases}$$

Now, let $u_{01}=[-0.001,0,0.001]$, $u_{21}=[10.999,11,11.001]$, $u_{10}=[3.999,4,4.001]$. and $u_{12}=[6.997,7,7.003]$. Firstly, fuzzy membership function for u_{01} is

$$\mu_{u01}(x)=\begin{cases}\frac{x+0.001}{0.001}, & -0.001\le x\le 0\\ \frac{0.001-x}{0.001}, & 0\le x\le 0.001\end{cases}$$

and $[u_{01}]_\alpha=[0.001\alpha-0.001, 0.001-0.001\alpha]$.
Then, fuzzy membership function for u_{21} is

$$\mu_{u21}(x)=\begin{cases}\frac{x-10.999}{0.001}, & 10.999\le x\le 11\\ \frac{11.001-x}{0.001}, & 11\le x\le 11.001\end{cases}$$

and $[u_{21}]_\alpha=[0.001\alpha+10.999, 11.001-0.001\alpha]$.
Then, fuzzy membership function for u_{10} is

$$\mu_{u10}(x)=\begin{cases}\frac{x-3.999}{0.001}, & 3.999\le x\le 4\\ \frac{4.001-x}{0.001}, & 4\le x\le 4.001\end{cases}$$

and $[u_{10}]_\alpha=[0.001\alpha+3.999, 4.001-0.001\alpha]$.
Thus, $[u_{12}]_\alpha=[0.003\alpha+6.997, 7.003-0.003\alpha]$.
Let $X_1=6.997+0.003\alpha\Rightarrow\alpha=\frac{X_1-6.997}{0.003}$ and
$X_2=7.003-0.003\alpha\Rightarrow\alpha=\frac{-X_2+7.003}{0.003}$.
Hence, fuzzy membership function for u_{12} is

$$\mu_{u12}(x)=\begin{cases}\frac{x-6.997}{0.003}, & 6.997\le x\le 7\\ \frac{7.003-x}{0.003}, & 7\le x\le 7.003\end{cases}$$

Similarly, fuzzy membership functions can be found out at different grid points. In Table 1-2, the α-cut with left and right functions of fuzzy number u is shown for $\alpha=0$, respectively.

6. CONCLUSION

In this paper, fuzzy membership functions to find solution of wave equation has been shown in numerical fuzzified form. Numerical computations have been made to illustrate ability and reliability of the method.

Table 1. The left side of fuzzy number u for α=0.

j\\i	0	1	2	3	4	5
0	-0.001	3.999	11.999	17.999	15.999	-0.001
1	-0.001	5.999	10.999	13.999	8.999	-0.001
2	-0.001	6.997	7.997	1.997	-2.003	-0.001
3	-0.001	1.995	-2.007	-8.007	-7.005	-0.001
4	-0.001	-9.011	-14.015	-11.015	-6.011	-0.001
5	-0.001	-16.021	-18.033	-12.033	-4.021	-0.001
6	-0.001	-9.035	-14.069	-11.069	-6.045	-0.001
7	-0.001	1.909	-2.137	-8.147	-7.091	-0.001
8	-0.001	6.817	7.693	1.703	-2.193	-0.001
9	-0.001	5.601	10.373	13.353	8.611	-0.001
10	-0.001	3.189	10.647	16.677	15.159	-0.001

Table 2. The right side of fuzzy number u for α=0.

j\\i	0	1	2	3	4	5
0	0.001	4.001	12.001	18.001	16.001	0.001
1	0.001	6.001	11.001	14.001	9.001	0.001
2	0.001	7.003	8.003	2.003	-1.997	0.001
3	0.001	2.005	-1.993	-7.993	-6.995	0.001
4	0.001	-8.989	-13.985	-10.985	-5.989	0.001
5	0.001	-15.979	-17.967	-11.967	-3.979	0.001
6	0.001	-8.955	-13.931	-10.931	-5.955	0.001
7	0.001	2.091	-1.853	-7.853	-6.909	0.001
8	0.001	7.183	8.307	2.307	-1.807	0.001
9	0.001	6.399	11.627	14.647	9.399	0.001
10	0.001	5.811	13.353	19.323	15.841	0.001

7. REFERENCES

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