

Square Divisor Cordial Graphs

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ABSTRACT

A square divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \dots, |V|\}$ such that if each edge uv is assigned the label 1 if $[f(u)]^2 | f(v)$ or $[f(v)]^2 | f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a square divisor cordial labeling, then it is called square divisor cordial graph. In this paper, we investigate the square divisor cordial labeling behaviour of paths, cycles, wheel graphs, star graphs, some complete bipartite graphs and complete graphs.

Keywords:

Cordial labeling, square divisor cordial labeling, square divisor cordial graphs.

1. INTRODUCTION

By a graph G , we mean a finite, undirected graph without loops and multiple edges, for terms not defined here, we refer to Harary [5].

First we give the concept in number theory [3].

Definition 1. Let a and b be two integers. We say that a divides b if there is a positive integer k such that $b = ka$. It is denoted by $a|b$.

If a does not divide b , then we denote $a \nmid b$.

Graph labeling [4] is a strong communication between number theory [3] and structure of graphs [5]. By combining the divisibility concept in number theory and cordial labeling concept in graph labeling, we introduce a new concept called square divisor cordial labeling. In this paper, we prove the standard graphs such as paths P_n ($n \leq 12$), cycles C_n ($3 \leq n \leq 11$), wheels, some star graphs, some complete bipartite graphs and some complete graphs are square divisor cordial.

A vertex labeling [4] of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. The two best known labeling methods are graceful and harmonious labelings. Cordial labeling is a variation of both graceful and harmonious labelings [1].

Definition 2. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f .

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of

edges having labels 0 and 1 respectively under f^* .

The concept of cordial labeling was introduced by Cahit [1].

Definition 3. A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

Cahit proved some results in [2].

2. MAIN RESULTS

Sundaram, Ponraj and Somasundaram [6] have introduced the notion of prime cordial labeling.

Definition 4. [6] A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \dots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1.

In [6], they have proved some graphs are prime cordial.

Varatharajan, Navaneethakrishnan and Nagarajan [7] have introduced the notion of divisor cordial labeling.

Definition 5. [7] Let $G = (V, E)$ be a simple graph and $f : V \rightarrow \{1, 2, \dots, |V|\}$ be a bijection. For an edge uv , assign the label 1 if either $f(u)|f(v)$ or $f(v)|f(u)$ and the label 0 if $f(u) \nmid f(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.

A graph with a divisor cordial labeling is called a divisor cordial graph.

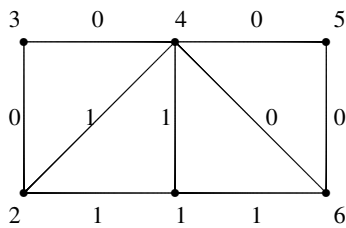
In [7], they have proved the standard graphs such as path, cycle, wheel, star and some complete bipartite graphs are divisor cordial. They also proved that complete graph is not divisor cordial. In [8], they have proved some special classes of graphs such as dragon, corona, full binary trees, $G * K_{2,n}$ and $G * K_{3,n}$ are divisor cordial. Motivated by the concept of divisor cordial labeling, we introduce a new special type of cordial labeling called square divisor cordial labeling as follows.

Definition 6. Let $G = (V, E)$ be a simple graph and $f : V \rightarrow \{1, 2, \dots, |V|\}$ be a bijection. For each edge uv , assign the label 1 if either $[f(u)]^2 | f(v)$ or $[f(v)]^2 | f(u)$ and the label 0 otherwise. f is called a square divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.

A graph with a square divisor cordial labeling is called a square divisor cordial graph.

Example 1.

Consider the following graph G .



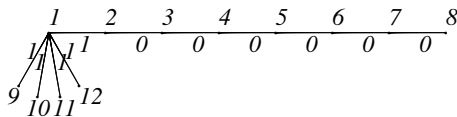
We see that $e_f(0) = 5$ and $e_f(1) = 4$. Thus $|e_f(0) - e_f(1)| = 1$ and hence G is square divisor cordial.

Theorem 1. Given a positive integer n , there is a square divisor cordial graph G which has n vertices.

PROOF. Suppose n is even. By constructing a path containing $\frac{n}{2} + 2$ vertices $v_1, v_2, \dots, v_{\frac{n}{2}+2}$ which are labeled as $1, 2, \dots, \frac{n}{2} + 2$ respectively and attaching $\frac{n}{2} - 2$ vertices $v_{\frac{n}{2}+3}, v_{\frac{n}{2}+4}, \dots, v_n$ which are labeled as $\frac{n}{2} + 3, \frac{n}{2} + 4, \dots, n$ respectively, to the vertex v_1 , we see that $e_f(0) = \frac{n}{2}$ and $e_f(1) = \frac{n}{2} - 1$ and hence $|e_f(0) - e_f(1)| \leq 1$. Thus the resultant graph G is square divisor cordial. Similarly we can construct a graph for n is odd.

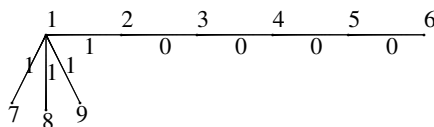
Theorem 2.5 can be illustrated in the following example.

Example 2. (1) n is even. Let $n = 12$.



Here $e_f(0) = 6$ and $e_f(1) = 5$. Hence $|e_f(0) - e_f(1)| = 1$.

(2) n is odd. Let $n = 9$.



Here $e_f(0) = e_f(1) = 4$ and $|e_f(0) - e_f(1)| = 0$.

Theorem 2. If G is a square divisor cordial graph of even size, then $G - e$ is also square divisor cordial for all $e \in E(G)$.

PROOF. Let q be the even size of the square divisor cordial graph G . Then it follows that $e_f(0) = e_f(1) = \frac{q}{2}$. Let e be any edge in G which is labeled either 0 or 1. Then in $G - e$, we have either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$ and hence $|e_f(0) - e_f(1)| \leq 1$. Thus $G - e$ is square divisor cordial.

Theorem 3. If G is a square divisor cordial graph of odd size, then $G - e$ is also square divisor cordial for some $e \in E(G)$.

PROOF. Let q be the odd size of the square divisor cordial graph G . Then it follows that either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$. If $e_f(0) = e_f(1) + 1$, then remove the edge e which is labeled as 0 and if $e_f(1) = e_f(0) + 1$, then remove the edge e which is labeled 1 from G . then it follows that $e_f(0) = e_f(1)$. Thus $G - e$ is square divisor cordial for some $e \in E(G)$.

Theorem 4. The path P_n is square divisor cordial if and only if $n \leq 12$.

PROOF. Let $P_n : v_1 v_2 \dots v_n$ be the path.

Case(i): $n \leq 12$

The following table gives the square divisor cordial labeling of P_n , $n \leq 12$.

Vertex labels:

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
1	1											
2	1	2										
3	1	2	3									
4	1	2	4	3								
5	1	2	4	3	5							
6	4	2	1	3	5	6						
7	4	2	1	3	5	6	7					
8	4	2	8	1	3	5	6	7				
9	4	2	8	1	3	5	6	7	9			
10	4	2	8	1	3	9	5	6	7	10		
11	4	2	8	1	3	9	5	6	7	10	11	
12	4	2	8	1	3	9	5	6	7	10	11	12

Case(ii): $n \geq 13$

If possible, let there be a square divisor cordial labeling f .

Suppose n is odd. Then obviously $e_f(i) = \frac{n-1}{2}$, $i = 0, 1$.

In order to get the labels 1, atmost two numbers can be assigned to the vertices that are adjacent to the vertices labeled with $1, 2, \dots, \lfloor \sqrt{n} \rfloor - 1$ and atmost one number can be assigned to the vertex that is adjacent to the vertex labeled with $\lfloor \sqrt{n} \rfloor$. Therefore we have

$$e_f(1) \leq 2(\lfloor \sqrt{n} \rfloor - 1) + 1 \\ \Rightarrow e_f(1) \leq 2 \lfloor \sqrt{n} \rfloor - 1$$

Hence

$$e_f(0) \geq n - 1 - (2 \lfloor \sqrt{n} \rfloor - 1) \\ = n - 2 \lfloor \sqrt{n} \rfloor \\ > \frac{n-1}{2}, \quad \text{a contradiction.}$$

Thus if n is odd, there can not be a square divisor cordial labeling.

Similarly if n is even, either $e_f(0) = \frac{n}{2}$ and $e_f(1) = \frac{n}{2} - 1$ or $e_f(1) = \frac{n}{2}$ and $e_f(0) = \frac{n}{2} - 1$ and hence there can not be a square divisor cordial labeling.

Theorem 5. The cycle C_n is square divisor cordial if and only if $3 \leq n \leq 11$.

PROOF. Let $C_n : v_1 v_2 \dots v_n v_1$ be the cycle.

Case(i): $3 \leq n \leq 11$

The square divisor cordial labeling of C_n , $3 \leq n \leq 11$ is given below.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
3	1	2	3								
4	1	2	3	4							
5	1	2	4	3	5						
6	1	2	4	3	5	6					
7	1	2	4	3	5	6	7				
8	1	3	5	6	7	4	2	8			
9	1	3	5	6	7	9	4	2	8		
10	1	3	9	5	6	7	10	4	2	8	
11	1	3	9	5	6	7	10	11	4	12	8

Case(ii): $n \geq 12$

If there is a square divisor cordial labeling f , then proceeding as in the case of path, we get contradiction.

Theorem 6. The wheel graph $W_n = K_1 + C_{n-1}$ is square divisor cordial.

PROOF. Let v_1 be the central vertex and v_2, v_3, \dots, v_n be the vertices of C_{n-1} . Define

$$f(v_1) = 1$$

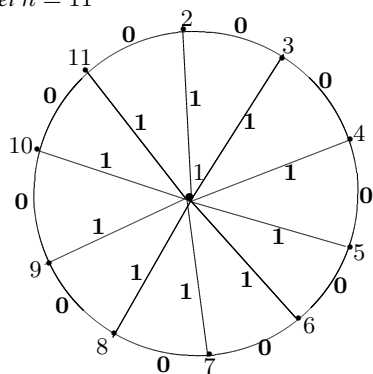
and

$$f(v_i) = i, 2 \leq i \leq n$$

We see that $e_f(0) = e_f(1) = n - 1$. Hence $|e_f(0) - e_f(1)| = 0$. Thus W_n is square divisor cordial.

Theorem 2.11 can be illustrated in the following example.

Example 3. Let $n = 11$



We see that $e_f(0) = e_f(1) = 10$.

Theorem 7. The star graph $K_{1,n}$ is square divisor cordial if and only if $n = 2, 3, 4, 5$ or 7 .

PROOF. Let v be the apex vertex and let v_1, v_2, \dots, v_n be the pendant vertices of the star $K_{1,n}$.

Case(i): $n = 2, 3, 4, 5$ or 7

Assign the label 2 to the vertex v and the remaining labels to the vertices v_1, v_2, \dots, v_n . If n is even, then $e_f(0) = e_f(1)$ and if n is odd, then either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$. Thus $|e_f(0) - e_f(1)| \leq 1$ and hence $K_{1,n}$ is square divisor cordial.

Case(ii): $n = 6$

By labeling the vertices as in case(i), we see that $|e_f(0) - e_f(1)| = 2$ and hence $K_{1,n}$ is not square divisor cordial.

Case(iii): $n \geq 8$

By labeling the vertices as in case(i), we see that $|e_f(0) - e_f(1)| \geq 2$ and hence $K_{1,n}$ is not square divisor cordial.

Theorem 8. The complete bipartite graph $K_{2,n}$ is square divisor cordial.

PROOF. Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{x_1, x_2\}$ and $V_2 = \{y_1, y_2, \dots, y_n\}$. Now assign the label 1 to x_1 , the largest prime number p such that $p \leq n + 2$ to x_2 and the remaining labels to the vertices y_1, y_2, \dots, y_n . Then it follows that $e_f(0) = e_f(1) = n$ and hence $K_{2,n}$ is square divisor cordial.

Theorem 9. The complete bipartite graph $K_{3,n}$ is square divisor cordial if and only if $n = 1, 2, 3, 5, 6, 7$ or 9 .

PROOF. Proof: Let $V = V_1 \cup V_2$ be the bipartition of $K_{3,n}$ such that $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, \dots, y_n\}$.

Case(i): $n = 1, 2, 3$ or 5

Assign the label 1 to x_1 , 2 to x_2 , the largest prime number p such that $p \leq n + 3$ to x_3 and the remaining labels to the vertices y_1, y_2, \dots, y_n . If n is even, then $e_f(0) = e_f(1)$ and if n is odd, then either $e_f(0) = e_f(1) + 1$ or $e_f(1) = e_f(0) + 1$. Thus $|e_f(0) - e_f(1)| \leq 1$ and hence $K_{3,n}$ is square divisor cordial.

Case(ii): $n = 6, 7$ or 9

Label the vertices as in case(i) and interchange the labels p and 9. If n is even, then $e_f(0) = e_f(1)$ and if n is odd, then $e_f(0) = e_f(1) + 1$. Thus $|e_f(0) - e_f(1)| \leq 1$ and hence $K_{3,n}$ is square divisor cordial.

Case(iii): $n = 4$

By labeling the vertices as in case(i), we see that $|e_f(0) - e_f(1)| = 2$ and hence $K_{3,n}$ is square not divisor cordial.

Case(iv): $n = 8$ and $n \geq 10$

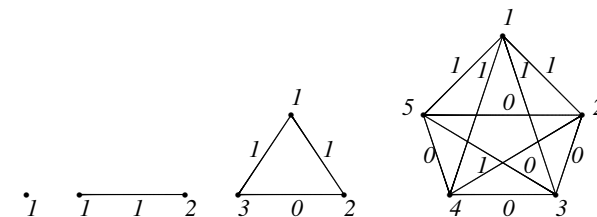
By labeling the vertices as in case(ii), we see that $|e_f(0) - e_f(1)| \geq 2$ and hence $K_{3,n}$ is square not divisor cordial.

Theorem 10. The complete graph K_n is square divisor cordial if and only if $n = 1, 2, 3$ or 5 .

PROOF. Let v_1, v_2, \dots, v_n be the vertices of K_n with $f(v_i) = i$.

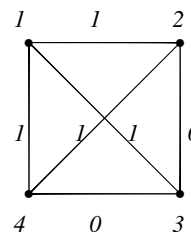
Case(i): $n = 1, 2, 3$ or 5

The square divisor cordial labeling of K_1, K_2, K_3 and K_5 are given as follows.



Case(ii): $n = 4$

The labeling of K_4 is given as follows.



We see that $|e_f(0) - e_f(1)| = 2$ and hence K_4 is not square divisor cordial.

Case(iii): $n \geq 6$

If possible, let there be a square divisor cordial labeling f of K_n . Suppose $n \equiv 0, 1 \pmod{4}$.

Then obviously $e_f(i) = \frac{n(n-1)}{4}$, $i = 0, 1$.

Since v_1 contributes $n - 1$, v_2 contributes $\lfloor \frac{n}{2} \rfloor$, v_3 contributes

$\lfloor \frac{n}{3^2} \rfloor, \dots$ to $e_f(1)$, we have

$$e_f(1) = n - 1 + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \dots + \left\lfloor \frac{n}{k^2} \right\rfloor,$$

where k is the largest positive integer such that $k^2 \leq n$.
Since $n \geq 6$,

$$\begin{aligned} \frac{(n-1)(n-4)}{4} &> \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \dots + \left\lfloor \frac{n}{k^2} \right\rfloor \\ \Rightarrow \frac{n(n-1)}{4} - (n-1) &> \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \dots + \left\lfloor \frac{n}{k^2} \right\rfloor \\ \Rightarrow \frac{n(n-1)}{4} &> n-1 + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \dots + \left\lfloor \frac{n}{k^2} \right\rfloor \\ \Rightarrow e_f(1) &> n-1 + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \dots + \left\lfloor \frac{n}{k^2} \right\rfloor, \end{aligned}$$

a contradiction.

The other cases namely $n \equiv 2, 3 \pmod{4}$ can be dealt with on similar lines.

3. CONCLUDING REMARKS

As not every divisor cordial graph is square divisor cordial it is very interesting to investigate graphs which admit square divisor cordial labeling. We have tried to investigate square divisor cor-

dial labeling of standard graphs. To investigate analogous results for different graphs is an open area of research.

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