Some Separation Properties of the Digital Line

Nirmala Mariappan
Department of Mathematics
Lady Doak College, Madurai
Tamilnadu, India

M. Lellis Thivagar
Department of Mathematics
Madurai Kamaraj University
Madurai, Tamilnadu, India

ABSTRACT

This paper deals with some separation properties of the digital line, semi-regularity, semi-normality, T_b separation and α T_b separation of the digital line.

Key words and phrases:

digital line, semi-open sets, g-closed sets,

ω-closed sets, $\alpha\hat{g}$-closed sets, semi-regular space, semi-normal space.

1. INTRODUCTION

Levine [15], Mashhour et al [18] and Njastad [21] introduced semi-open sets, preopen sets and α open sets respectively. Levine [16] introduced g-closed sets and studied their most fundamental properties. P.Bhattacharya and B.K.Lahiri [3], S.P.Arya and T.Nour [2], H.Maki et al [19] introduced sg-closed sets, gs-closed sets, α g-closed sets and gα-closed sets respectively. P.Sundaram and M.Sheik John [24] introduced and studied ω-closed sets.

A characterization of semi-open sets in the digital line is given in this paper. Some separation properties of the digital line are also studied.

2. SOME SEPARATION PROPERTIES OF (Z, κ)

The digital line or so called the Khalimsky line is the set of integers Z, equipped with the topology κ generated by $\{\{2m-1, 2m, 2m+1\} / m \in Z\}$. The concept of the digital line (Z, κ) is initiated by Khalimsky [10],[11]. This topological space is denoted by (Z, κ). In (Z, κ) each singleton {2n} is closed and each singleton {2n+1} is open where $n \in Z$. If U(x) is the smallest open set containing x, then U(2m) = {2m-1, 2m, 2m+1} and U(2m+1) = {2m +1} where $m \in Z$. It is well known that (Z, κ) is $T_{1/2}$ and $T_{3/4}$ but it is not T_1. In the present section, the semi-regulality and semi-normality of the digital line are proved and so an alternative proof of [8,Theorem B] is given. Theorem B [8] shows the digital line is s-normal.

Lemma 2.1 ([14], lines 12-13 in page 175)

For a subset A of (Z, κ) to be open it is necessary and sufficient that $2m \pm 1 \in A$ whenever $2m \in A$.

Proof Necessity. Let $2m \in A$. Since A is open, U(2m) = $\{2m-1, 2m, 2m+1\} \subseteq A$. Sufficiency. To prove that A = int(A). Let $x \in A$. Case 1. x = 2m. By the hypothesis $2m \pm 1 \in A$ and therefore $U(2m) \subseteq A$. This implies $x \in$ int(A). Case 2.

x = 2m+1. Since {2m + 1} is an open subset of Z, $x \in$ int(A).

Lemma 2.2

A subset A of (Z, κ) is not closed if and only if there exists $2m+1 \in A$ such that 2m or $2m+2 \notin A$.

Proof Necessity. A is not closed implies A^c is not open. Therefore, by Lemma 2.1, there exists $2m \in A^c$ such that 2m+1 or $2m-1 \notin A^c$. Case 1. $2m+1 \notin A^c$. Then $2m+1 \in A$ and $2m \notin A$. Case 2. $2m-1 \notin A^c$. Then $2m-1 \in A$ and $2m \notin A$. Thus there exists $2m+1 \in A$ such that 2m or $2m+2 \notin A$.

Sufficiency. Let there exist $2m+1 \in A$ such that 2m or $2m+2 \notin A$. Then 2m or $2m+2 \in A^c$ and $2m+1 \notin A^c$. Therefore by Lemma 2.1, A^c is not open and hence A is not closed.

Theorem 2.3

A subset A of (Z, κ) is semi-closed if and only if 2n-1 or $2n+1 \notin A$ whenever $2n \notin A$.

Proof Necessity. Let $A \subseteq Z$ be semi-closed and $2n \notin A$. Suppose 2n-1 and $2n+1 \in A$.

Then $cl(\{2n-1, 2n+1\}) \subseteq cl(A)$. This implies $int(cl(\{2n-1, 2n+1\})) \subseteq int(cl((A)) \subseteq A$. That is $\{2n-1, 2n, 2n+1\} \subseteq A$, which implies $2n \in A$, a contradiction. Sufficiency. To prove that $A \supseteq int(cl(A))$. Let $x \in int(cl(A))$.

Case 1. x = 2m+1. $x \in int(cl(A)) \subseteq cl(A)$. Since {x} is open, $x \in A$.

Case 2. x = 2m. $x \in int(cl(A))$ implies

$\{2m-1, 2m, 2m+1\} \subseteq cl(A)$. Since {2m-1}and {2m+1}are open, 2m-1 and $2m+1 \in A$. Then, by assumption $2m \in A$.

Theorem 2.4

A subset A of (Z, κ) is semi-open if and only if 2n-1 or 2n+1 $\in$ A whenever 2n $\in$ A.

Proof Necessity. Let A $\subseteq$ Z be semi-open and 2n $\in$ A. Suppose 2n-1 and 2n+1 $\notin$ A. Then

int(A) $\cap$ {2n-1, 2n, 2n +1}= ϕ . This implies int(A) $\subseteq G^c$ where G ={2n-1, 2n, 2n+1}is open. This implies cl(int(A)) $\subseteq G^c$ and therefore 2n $\notin$ cl(int(A)) or

A $\not\subset$ cl(int(A)), a contradiction.

Sufficiency. To prove that A $\subseteq$ cl(int(A)). Let x $\in$ A. Case 1. x = 2m+1. Then x $\in$ int(A) and therefore x $\in$ cl(int(A)). Case 2. x = 2m. Then 2m-1 or 2m+1 $\in$ A. This implies 2m-1 or 2m+1 $\in$ int(A) and therefore 2m $\in$ cl({2m-1}) $\subseteq$ cl(int(A)) or 2m $\in$ cl({2m+1}) $\subseteq$ cl(int(A)).

Using the characterization of the semi-closed subsets of (Z, κ), it can be proved that (Z, κ) is semi-regular and semi-normal.

Theorem 2.5

(Z, κ) is semi-regular.

Proof Let A be a semi-closed subset of (Z, κ) and x $\notin$ A. Case 1. x = 2n. Since x = 2n $\notin$ A, by Theorem 2.3, 2n-1 or 2n+1 $\notin$ A. Let U ={2n-1,2n} if 2n-1 $\notin$ A and U = {2n, 2n + 1}if 2n+1 $\notin$ A. Let V = Z - U. Case 2. x = 2n+1. Let U ={2n + 1} and V = Z - U. In each case U and V are disjoint semi-open sets such that x $\in$ U and A $\subseteq$ V . Hence (Z, κ) is semi-regular.

Theorem 2.6

(Z, κ) is semi-normal.

Proof Let A and B be disjoint semi-closed subsets of (Z, κ). Let A = $O_1 \cup E_1$ and B = $O_2 \cup E_2$ where O_1 and O_2 are subsets of 2Z + 1 and E_1 and E_2 are subsets of 2Z. Let us form the semi-open sets U and V as follows. Let 2n $\in E_1$. Then 2n $\notin E_2$ and therefore 2n $\notin$ B. Since B is semi-closed 2n-1 or 2n+1 $\notin$ B. Let $D_1 = \bigcup_{2n\in E_1, x=2n\pm1, x\notin B} \{2n, x\}$ and U = $O_1 \cup D_1$. Similarly let V = $O_2 \cup D_2$ where

$D2 = \bigcup_{2m\in E_2, x=2m\pm1, x\notin A} \{2m, x\}$. Then U and V are semi-open subsets of (Z, κ) containing A and B respectively. Also U $\cap$ V = ϕ .

Now some more separation properties of (Z, κ).

Let us recall the following definitions.

Definition 2.7

A subset A of a topological space (X, τ) is called

i. generalized closed (briefly g-closed) [16] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

ii. semi-generalized closed (briefly sg-closed) [3] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X.

iii. generalized semi-closed (briefly gs-closed) [3] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

iv. α -generalized closed (briefly α g-closed) [19] if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

v. generalized α -closed (briefly g α -closed) [19] if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is α -open in X.

vi. ω -closed or $\hat{g}$ -closed [24] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X.

Since any singleton subset of (Z, κ) is open or closed, (Z, κ) is $T_{1/2}$ [7]. Therefore the class of g-open sets = the class of open sets in (Z, κ). Hence from Definition 3.1, it follows that, the class of ω -closed sets = the class of closed sets.

Since (Z, κ) is $T_{3/4}$ and $T_{3/4} = T_{gs}$ + semi T_1 [4], (Z, κ) is also T_{gs}. Hence in (Z, κ), the class of gs-open sets = the class of sg-open sets = the class of semi-open sets since (Z, κ) is semi-$T_{1/2}$.

Definition 2.8

A subset A of a topological space (X, τ) is called α - $\hat{g}$ -closed (briefly α $\hat{g}$ -closed) [1], if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\hat{g}$ -open Since $\hat{g}$ -open sets in (Z, κ) are open sets, a subset A of Z is α $\hat{g}$ -closed if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in Z. That is the class of α $\hat{g}$ -closed sets in (Z, κ) = the class of α g-closed sets in (Z, κ). In (Z, κ),

PO(Z, κ) = κ = α O(Z, κ).

Hence the class of α $\hat{g}$ -closed sets in (Z, κ)

= the class of α g-closed sets in (Z, κ)

= the class of g-closed sets in (Z, κ)

= the class of closed sets in (Z, κ),

since (Z, κ) is $T_{1/2}$.

Also the class of g α -closed sets in (Z, κ)

= the class of closed sets in (Z, κ).

Definition 2.9

A topological space (X, τ) is called $T_{\alpha\hat{g}}$ -space if every α $\hat{g}$ -closed set is α -closed.

Since α -closed sets in (Z, κ) are the closed sets in (Z, κ), (Z, κ) is a $T_{\alpha\hat{g}}$ -space.

Definition 2.10

A topological space (X, τ) is called a T_b-space if every gs-closed set in X is closed.

(Z, κ) is not a T_b-space. The set A = {2n-1, 2n} is semi-closed and therefore gs-closed in (Z, κ). But A is not closed in (Z, κ).

Definition 2.11

A topological space (X, τ) is called a α T_b-space if every α g-closed set in X is closed.

(Z, κ) is a α T_b-space.

REFERENCES

[1] M.E. Abd El-Monsef, S. Rose Mary and M. Lellis Thivagar, 2007, $\alpha\ \hat{g}$ -closed sets in topological spaces, Assiut Univ. J. of Mathematics and Computer Science,36(1)43-51.

[2] S.P. Arya and T. Nour, 1990, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21(8), 717-719.

[3] P. Bhattacharya and B.K. Lahiri, 1987, Semi-generalised closed sets in topology, Indian J. Math., 29, 375-382.

[4] J. Dontchev and M. Ganster, 1996, On δ generalized closed sets and $T_{3/4}$ -spaces, Mem. Fac. Sci. Kochi Univ.(math),17,15-42.

[5] C. Dorsett, 1982, Semi-regular spaces, Soochow J. Math.8, 45-53.

[6] C. Dorsett, 1985, Semi-normal spaces, Kyungpook Math. J25, 173-180.

[7] W. Dunham, 1977, $T_{1/2}$-spaces, Kyungpook Math. J. 17,161-169.

[8] M. Fujimoto, S. Takigava, J. Dontchev, T. Noiri and H. Maki, 2006, The topological structures and groups of digital n-spaces, Kochi J. Math., 1, 31-55.

[9] T. Fukutake, P. Sundaram and M. Sheik John, 2002, ω-ciosed sets, ω-open sets and ω-continuity in bitopological spaces, Bull. Fukuoka Univ. Ed. Part III, 51,1-9.

[10] E.D. Khalimsky, 1970, Applications of connected ordered topological spaces in topology, Conference of math. departments of Povolsia.

[11] E.D. Khalimsky, 1987, Topologicals structures in computer sciences, J . Appl. Math., Simulation, 1(1) , 25-40.

[12] E.D. Khalimsky, R. Kopperman and P. R. Meyer, 1990, Computer graphics and connected topologies on finite ordered sets, Topology Appl., 36,1-17.

[13] T. Y. Kong, R. Kopperman and P. R. Meyer, 1991, A topologicai approach to digital topology, Amer. Math. Monthly, 98, 901- 917.

[14] V. Kovalevsky and R. Kopperman, 1994, Some topology-based image processing algorithms, Annals of the New York Academy of Sciences, Papers on General Topology and its Applications, 728, 174-182.

[15] N. Levine, 1963, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70, 36-41.

[16] N. Levine, 1970, Generalised closed sets in topology, Rend. circ. Mat. Palermo, 19, 89-96.

[17] S. Maheshwari and R. Prasada, 1978, On s-normai spaces, Bull. Math. Soc. Sci. R.S. Roumanie, 22(70) (1), 27-29.

[18] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El Deeb, 1982, On pre continuous and weak pre continuous mappings, Proc. Math. Phys. Soc. Egypt., 53, 47-53.

[19] H. Maki, R. Devi and K. Balachandran, 1994, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ.Ser.A.Math.15, 51-63.

[20] S.I. Nada, 2004, Semi-open and semi-closed sets in digital toprological spaces, Commun. Fac. Sci. Uniu. Ank. Series A1, 53(1), 1-6.

[21] O. Njastad, 1965, On some classes of nearly open sets, Pacific J.Math.15, 961- 970.

[22] M. Sheik John, 2000, A study on generalizations of closed sets and continuous maps in topoiogical spaces, Ph.D. Thesis, Bharathiar University, Coimbatore, India.

[23] P. Sundaram and M. Sheik John, 1995, Weakly closed sets weak continuous maps in topological spaces, Proc.82 nd Indian Sci. Cong. Calcutta, 49.

[24] P. Sundaram ancl M. Sheik John, 2000, On ω-closed sets in topology, Acta ciencia Indica, 26M (4), 389-392.