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On Properties of Fuzzy Mealy Machines

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ABSTRACT

It has been shown that the problem of equivalent and minimization of fuzzy Mealy machines can be resolved via their algebraic study. However, no attention has paid to study fuzzy Mealy machines topologically. This paper introduces topology on the state set of a fuzzy Mealy machine and study of various kinds of fuzzy Mealy machines viz. cyclic, retrievable, strongly connected, with exchange property and connected through this topology. In addition, various products of fuzzy Mealy machines and their relationship in regards to aforementioned kinds of fuzzy Mealy machines are also studied.

Keywords:

Fuzzy finite state machines, Fuzzy Mealy machines

1. INTRODUCTION

Recently, there has been tremendous growth in research on fuzzy automata theory, theoretically[8-10,12] as well as practically [5,7,10,13]. Fuzzy Mealy machine is a kind of fuzzy automaton with outputs capabilities based on both the current state and the current input on similar lines to fuzzy automaton, fuzzy Mealy machine generalize classical Mealy machines, in the sense of partial (degree of) transition of states and outputs.This makes possible to tackled uncertainty in transition as well output. Surprisingly, very little research has done in this area [2,4,11]. Algebraic study of fuzzy Mealy machines has done mainly by Mordeson et al. [11] and Jun Liu, Zhiwen Mo, Dong Qiu and Yang Wang [4]. Both the treatments study internal working of fuzzy Mealy machines such as equivalence and minimization. This kind of study for finite state machines has been discussed by Mordeson et. al. [8] and Kumbhojkar and Chaudhari[6]. In [1,3] equivalence of fuzzy Mealy machine and fuzzy Moore machine is discussed. The aim of the present paper is to study fuzzy Mealy machine with the help of the topology induced by the successor operator defined on its state set. Here, submachines, (strongly) connected, cyclic (singly generated) and retrievable fuzzy Mealy machines with the help of this topology are studied. This study is analogous to that of topological study of fuzzy finite state machines discussed in [9]. Also products of fuzzy Mealy machines and their cyclicness, retrievablity, strongly connectedness, exchange property and connectedness are discussed. Precisely following results are obtained in this paper
(i) Fuzzy Mealy machine M is strongly connected if and only if the topology generated by the successor function is a discrete topology. (ii) Fuzzy Mealy machine M satisfies the exchange property $\Leftrightarrow$ it is union of its strongly connected submachines $\Leftrightarrow$ it is retrievable (iii) Fuzzy Mealy machine M is strongly connected $\Leftrightarrow$ it is connected and retrievable $\Leftrightarrow$ its every submachine is strongly connected. It is also shown that the cartesian product and full direct product of fuzzy Mealy machines preserved these properties, where as the restricted direct product, cascade product and wreath product preserve these properties under strongly connectedness of individual fuzzy Mealy machine(s).
This paper consists of three sections. In section 2, successor function for the set of states of fuzzy Mealy machine is introduced. The concept of cyclic, retrievable, strongly connected, connected fuzzy Mealy machines along with their relationships are introduced. Sections 3, introduces various products of fuzzy Mealy machines such as cartesian product, full direct product, restricted direct product, cascade product, wreath product. The cyclicness, retrievability, and connectedness of these product are also discussed. The paper is concluded by giving future direction of research and by posing an open problem relating to the notion of topologies which are (always exists due to theorem (2.2)) for all the product introduced in this paper.

2. FUZZY MEALY MACHINES

Recall that a fuzzy Mealy machine (fmm) was introduced by Mordeson and Nair in [11] as a four tuple $M = (Q, X, Y, \mu)$,where Q is a finite non-empty set of states, X is a finite non-empty set of inputs symbols, Y is a finite non-empty set of output symbols and μ is a fuzzy subset of $Q \times X \times Q \times Y$ i.e. $\mu : Q \times X \times Q \times Y \longrightarrow [0, 1]$.
The extension of μ to $Q \times X^* \times Q \times Y^*$ is defined by

$$\mu^*(q, \lambda, p, \lambda) = \begin{cases} 1, & \text{if } q = p \\ 0, & \text{if } q \neq p, \end{cases}$$

$\mu^*(q, \lambda, p, b) = 0$ and $\mu^*(q, xa, p, yb) = \bigvee\{\mu^*(q, x, r, y) \wedge \mu(r, a, p, b) \mid r \in Q\}, \forall q, p \in Q, \forall x \in X^*, \forall a \in X, \forall b \in Y, \forall y \in Y^*$, where λ denotes the empty string.
Then $\forall q, p \in Q, \forall x, u \in X^*, \forall y, v \in Y^*$ such that $|x| = |y|$ and $|u| = |v|$, we have $\mu^*(q, xu, p, yv) = \bigvee\{\mu^*(q, x, r, y) \wedge \mu^*(r, u, p, v) \mid r \in Q\}$.
Therefore, $\forall q, p \in Q, \forall x \in X^*, \forall y \in Y^*$, if $|x| \neq |y|$ then $\mu^*(q, x, p, y) = 0$. Due to this property, for the rest of the paper we shall assume that $|x| = |y|$ in any expression of the form $\mu^*(q, x, p, y)$ that may encounter.
We now introduce a topology on the state set of a given fuzzy Mealy machine. We begin with the concept of successor.
Definition 2.1 Let $M = (Q, X, Y, \mu)$ be a fmm. Let $q, p \in Q$. Then p is called an *immediate successor* of q if $\exists\ a \in X$ and $b \in Y$ such that $\mu(q, a, p, b) > 0$ and p is called *successor* of q if $\exists\ x \in X^*$ and $y \in Y^*$ such that $\mu^*(q, x, p, y) > 0$.

Notation 2.2 Let $M = (Q, X, Y, \mu)$ be a fmm and $q \in Q$. We shall denote $S(q)$ the *set of all successor of* q.
Definition 2.3 Let $M = (Q, X, Y, \mu)$ be a fmm and $T \subseteq Q$. The *set of all successor of* T, denoted by $S_Q(T)$, is defined to be the set $S_Q(T) = \bigcup\{S(q) \mid q \in T\}$.
When no confusion arises, we shall write $S(T)$ for $S_Q(T)$
Theorem 2.1 Let $M = (Q, X, Y, \mu)$ be a fmm. Define a relation $\sim$ on Q as $p \sim q$ if and only if q is successor of p. Then $\sim$ is reflexive and transitive.
Theorem 2.2 Let $M = (Q, X, Y, \mu)$ be a fmm. Then the successor function, $S : \wp(Q) \longrightarrow \wp(Q)$, i.e. $S(A)$ the set of all successors of $A \subseteq Q$, is a Kuratowski's closure operator.
Proof. We prove idempotent property only. Clearly, $S(A) \subseteq S(S(A))$. Let $q \in S(S(A))$. Then $q \in S(p)$ for some $p \in S(A)$. Thus, $p \in S(r)$ for some $r \in A$. Now, q is successor of p and p is successor r. Thus, $q \in S(r) \subseteq S(A)$, that is, $S(S(A)) \subseteq S(A)$. Hence $S(S(A)) = S(A)$.
15mmTherefore, $\tau = \{A^c | S(A) \subseteq A\}$ defines a topology on Q. Thus, A subset A of Q is τ-closed, if $S(A) = A$.
Clearly the set of all closed subset of Q is a poset under set inclusion operation and it is also a complete lattice with $\wedge S(A_i) = \cap S(A_i)$ and $\vee S(A_i) = S(\cup(A_i))$.
Definition 2.4 Let $M = (Q, X, Y, \mu)$ be a fmm. Then $N = (T, X, Y, \nu)$ is called a *submachine of M,* if (1) T is τ-closed subset of Q and (2) $\mu_{|T\times X\times T\times Y} = \nu$. It is said to be proper submachine, if T is proper τ-closed subset of Q and $\mu_{|T\times X\times T\times Y} = \nu$.
Clearly, if K is a submachine of N and N is a submachine of M, then K is a submachine of M.
Definition 2.5 Let $M = (Q, X, Y, \mu)$ be a fmm. Let $q, p \in Q$ and $T \subseteq Q$. Suppose that if $p \in S(T \cup \{q\}), p \notin S(T)$, then $q \in S(T \cup \{p\})$. Then we say that M satisfies the exchange property.
Theorem 2.3 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M satisfies the exchange property if and only if $\forall p, q \in Q, q \in S(p)$ if and only if $p \in S(q)$.
Proof. Let $p, q \in Q$ and $p \in S(q)$. Now, $p \notin S(\phi)$, therefore $q \in S(p)$. On similar line $q \in S(p)$ implies that $p \in S(q)$.
Conversely let $T \subseteq Q, p, q \in Q$. Suppose $p \in S(T \cup \{q\}), p \notin S(T)$. Then $p \in S(q)$. Hence, $q \in S(p) \subseteq S(T \cup \{p\})$
Definition 2.6 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M is called *strongly connected* if $\forall p, q \in Q, p \in S(q)$
Theorem 2.4 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M is strongly connected if and only if τ is the discrete topology on Q.
Proof. Suppose M is strongly connected. Let $N = (T, X, Y, \nu)$ be a submachine of M such that $T \neq \phi$. Then $\exists\ q \in T$. Let $p \in Q$. Since M is strongly connected, $p \in S(q)$. Hence, $T = Q$ and so τ is discrete topology on Q.
Conversely if $p, q \in Q$ and $N = (S(q), X, Y, \nu)$, where $\nu = \mu_{|S(q)\times X\times S(q)\times Y}$, then N is a submachine of M. Since $S(q) \neq \phi$ and τ is discrete, we have $S(q) = Q$. Thus, $p \in S(q)$. Hence, M is strongly connected.
Theorem 2.5 Let $M = (Q, X, Y, \mu)$ be a fmm. Then for each $R \subseteq Q$, $S(R)$ is a τ-closed set.
Proof. $S(S(R)) = S(R)$ proves the theorem.
Thus, $N = (S(R), X, Y, \mu_R)$ is a submachine of M for each $R \subseteq Q$, where $\mu_R = \mu_{|S(R)\times X\times S(R)\times Y}$.
Definition 2.7 Let $M = (Q, X, Y, \mu)$ be a fmm. Let $R \subseteq Q$. The smallest submachine generated by R, i.e. $< R >$, is the intersection of all submachines of M whose state sets are subsets of Q containing R. Thus, $< R >= ((\bigcap_{i\in I} Q_i, X, Y, \bigcap_{i\in I} \mu_i)$, where $N_i = (Q_i, X, Y, \mu_i)$ is a submachine of M such that $R \subseteq Q_i, \forall i$.
Note that the state set of $< R >$ is the τ-closure of R.
Theorem 2.6 Let $M = (Q, X, Y, \mu)$ be a fmm. Let $R \subseteq Q$. Then $< R >= (S(R), X, Y, \mu_R)$.

Proof. Now, $< R >= (\bigcap_{i\in I} Q_i, X, Y, \bigcap_{i\in I} \mu_i)$, where $N_i = (Q_i, X, Y, \mu_i)$ are submachines of M such that $R \subseteq Q_i$. It suffices to show that $S(R) = \bigcap_{i\in I} Q_i$. Since $(S(R), X, Y, \mu_R)$ is a submachine of M such that $R \subseteq S(R)$, we have that $S(R) \supseteq \bigcap_{i\in I} Q_i$. Let $p \in S(R)$. Then $\exists r \in R$ and $x \in X^*, y \in Y^*$ such that $\mu^*(r, x, p, y) > 0$. Now $r \in \bigcap_{i\in I} Q_i$ and since $< R >$ is a submachine of M, $p \in \bigcap_{i\in I} Q_i$. Thus $S(R) \subseteq \bigcap_{i\in I} Q_i$. Hence $S(R) = \bigcap_{i\in I} Q_i$.
Definition 2.8 Let $M = (Q, X, Y, \mu)$ be a fmm. M is called *singly generated or cyclic* if $\exists\ q \in Q$ such that $\{q\}$ is τ-dense in Q. i.e $S(\{q\}) = Q$. In this case q is called a *generator* of M and we say that M is generated by q.
Hence, M is singly generated by $q \in Q$ if and only if $M = \langle\{q\}\rangle$.
Definition 2.9 A fmm $M = (Q, X, Y, \mu)$ is said to be *retrievable* when $\forall q \in Q, \forall x \in X^*, y \in Y^*$, if $\exists\ p \in Q$ such that $\mu^*(q, x, p, y) > 0$, then $\exists u \in X^*, v \in Y^*$ such that $\mu^*(p, u, q, v) > 0$.
Theorem 2.7 Let $M = (Q, X, Y, \mu)$ be a fmm. Then the following statements are equivalent

1. M satisfies the exchange property
2. M is union of strongly connected submachines
3. M is retrievable

Proof. (1) $\Rightarrow$ (2): Clearly $M = \bigcup_{i=1}^{n} < \{q_i\} >$, where $S(\{q_1, q_2, ..., q_n\}) = Q$. Also $S(q_i) \cap S(q_j) = \phi$ if $i \neq j$. Let $p, q \in S(q_i)$. Then $q_i \in S(p)$ and so $q \in S(p)$. Thus $< q_i >$ is strongly connected. (2) $\Rightarrow$ (1): Now $M = \bigcup_{i=1}^{n} M_i$, where each $M_i = (Q_i, X, Y, \mu_i)$ strongly connected. Let $p, q \in Q$. Suppose $p \in S(q)$. Now $\exists i$ such that $q \in Q_i$. Then $p \in S(q) \subseteq S(Q_i) = Q_i$. Thus $p, q \in Q_i$. Since M_i is strongly connected,$q \in S(p)$. Hence M satisfies the Exchange Property by theorem(2.3).
(2) $\Rightarrow$ (3): Now $M = \bigcup_{i=1}^{n} M_i$, where each $M_i = (Q_i, X, Y, \mu_i)$ strongly connected. Let $q \in Q, u \in X^*, v \in Y^*$ be such that $\mu^*(q, u, t, v) > 0$ for some $t \in Q$. Now $q \in Q_i$ for some i. Thus $t \in S(q) \subseteq S(Q_i)$. Since M_i is strongly connected, $q \in S(t)$. Hence $\exists x \in X^*, y \in Y^*$ such that $\mu^*(t, x, q, y) > 0$. Thus M is retrievable. (3) $\Rightarrow$ (2): Let $q \in Q$ and let $r, t \in S(q)$. Then $\exists x, u \in X^*$ and $\exists y, v \in Y^*$ such that $\mu^*(q, x, r, u) > 0$ and $\mu^*(q, u, r, v) > 0$. Since M is retrievable $\exists z \in X^*$ and $w \in Y^*$ such that $\mu^*(r, z, q, w) > 0$. Hence $q \in S(r)$ and $< q >$ is strongly connected. So $M = \bigcup_{q\in Q} < q >$.
Definition 2.10 Let $M = (Q, X, Y, \mu)$ be a fmm. A proper submachine $N = (T, X, Y, \nu)$ is said to be separated if Q-T is a proper τ-closed subset of Q.
Theorem 2.8 Let $M = (Q, X, Y, \mu)$ be a fmm. Let $N = (T, X, Y, \nu)$ be a submachine of M. Then N is *separated* if and only if $S(Q - T) \cap T = \phi$.
Proof. Suppose N is separated. Then T is τ-closed subset of Q. Thus, S(Q-T) = Q-T. Hence $S(Q - T) \cap T = Q - T \cap T = \phi$.
Conversely,let $q \in S(Q-T)$. Then by assumption $q \notin T$. Thus, $q \in Q - T$. Therefore, $S(Q - T) = Q - T$. i.e. $Q - T$ is a τ-closed subset of Q.
Theorem 2.9 Let $M = (Q, X, Y, \mu)$ be a cyclic then it has separated and strongly connected submachine.
Proof. Take $T = \{p \in Q \mid S(p) = Q\}$. Then $N = (T, X, Y, \mu\mid_{S(q)})$ is the required submachine.
Definition 2.11 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M is said to be *connected* if and only if Q has no proper τ-open and τ-closed subset.
Theorem 2.10 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M is connected if and only if M has no separated submachine.
Proof. Suppose M is connected. Let if possible M has a proper

separated submachine, say $N = (T, X, Y, \nu)$. Then T is a proper τ-closed subset of Q. Since, N is separated, we have $S(Q - T) = Q - T$. Therefore, Q_T is τ-closed. i.e. T is τ-open subset of Q. Therefore, T is proper τ-closed and τ–open subset of Q which is contradiction to M is connected.

Conversely, if M is not connected then M has a proper separated submachine, say $N = (T, X, Y, \nu)$. Then clearly, T is a proper τ-open and τ-closed subset of Q, which is contradiction to the hypothesis. Therefore, M is must not connected.

The difference between strongly connected and connected is depicted in the following example.

Example 2.11 Consider a Mealy machine $M : \mu(p, x_2, q, y_2) = 0.3, \mu(p, x_1, r, y_1) = 0.7$ and $\mu(q, x_1, r, y_1) = 0.8$.
Then M is connected but not strongly connected.

Theorem 2.12 Let $M = (Q, X, Y, \mu)$ be a fmm. Then M is connected if and only if for all proper submachines $N = (T, X, Y, \nu)$, $\exists\, s \in Q-T$ and $t \in T$ such that $S(s) \cap S(t) \neq \phi$.

Theorem 2.13 Let M be strongly connected fmm. Then M is cyclic, retrievable, connected.

Theorem 2.14 Let $M = (Q, X, Y, \mu)$ be a fmm. Then following assertions are equivalent.

1. M is strongly connected.
2. M is connected and retrievable.
3. Every submachine of M is strongly connected.

Proof. (1) $\Rightarrow$ (2): By theorem (2.4), M does not have any proper submachine and so M has no proper separated submachines. Thus M is connected. Now we show that M is retrievable. Let $q, t \in Q$ and $x \in X^*, y \in Y^*$ be such that $\mu^*(q, x, t, y) > 0$. Since M is strongly connected $q \in S(t)$. Then $\exists u \in X^*, v \in Y^*$ such that $\mu^*(t, u, q, v) > 0$. Hence M is retrievable.

(2) $\Rightarrow$ (3): Let $N = (T, X, Y, \nu)$ be a submachine of M. Suppose $p, q \in T$ are such that $p \notin S(q)$. Then $S(q) \neq Q$ and so $K = (S(q), X, Y, \mu \mid_{S(q)\times X\times S(q)\times Y})$ is a proper submachine of M. Since M is connected, $S(Q - S(q)) \cap S(q) \neq \phi$. Let $r \in S(Q - S(q)) \cap S(q)$. Then $r \in S(t)$ for some $t \in Q - S(q)$ and $r \in S(q)$. Now $\exists x \in X^* and y \in Y^*$ such that $\mu^*(t, x, r, y) > 0)$. Since M is retrievable, $\exists u \in X^*$ and $v \in Y^*$ such that $\mu^*(r, u, t, v) > 0)$. Thus $t \in S(r)$. Hence $t \in S(r) \subseteq S(q)$, a contradiction. Thus $p \in S(q) \forall p, q \in T$. Hence N is strongly connected.

(3) $\Rightarrow$ (1): Obvious.

Theorem 2.15 If M is cyclic and $T = \{p \in Q \mid S(p) = Q\}$ then $N = (T, X, Y, \mu_{|_{S(q)}})$ has a retrievable and connected submachine.

Proof. Follows by above theorem (2.13).

3. PRODUCTS OF FUZZY MEALY MACHINES

This section is an introduction of various products of fmms and discussed their interrelationship in terms of cyclicness, retrievability exchange property and connectedness.

Definition 3.1 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms,$i = 1, 2$. Then the machine $M_1 \odot M_2 = (Q, X, Y, \mu_1 \odot \mu_2)$ is called

1) the cartesian product of fmms of M_1 and M_2, with $X_1 \cap X_2 = \phi$ and $Y_1 \cap Y_2 = \phi$, if $Q = Q_1 \times Q_2, X = X_1 \cup X_2, Y = Y_1 \cup Y_2$ and $(\mu_1 \odot \mu_2)((q_1, q_2), a, (p_1, p_2), b) =$

$$= \begin{cases} \mu_1(q_1, a, p_1, b) & \text{if } a \in X_1, b \in Y_1 \text{ and } q_2 = p_2 \\ \mu_2(q_2, a, p_2, b) & \text{if } a \in X_2, b \in Y_2 \text{ and } q_1 = p_1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2, a \in X_1 \cup X_2, b \in Y_1 \cup Y_2$.

2) the full direct product of fmms of M_1 and M_2, if $Q = Q_1 \times Q_2, X = X_1 \times X_2, Y = Y_1 \times Y_2$ and $(\mu_1 \odot \mu_2)^*((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)) = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, \forall (x_1, x_2) \in X_1 \times X_2, \forall (y_1, y_2) \in Y_1 \times Y_2$.

3) the restricted direct product of fmms of M_1 and M_2, if $Q = Q_1 \times Q_2, X = X_1 = X_2, Y = Y_1 = Y_2$ and $(\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) = \mu_1^*(q_1, x, p_1, y) \wedge \mu_2^*(q_2, x, p_2, y)$, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, \forall x \in X, \forall y \in Y$.

4)the cascade product of fmms of M_1 and M_2, if $Q = Q_1 \times Q_2, X = X_2, Y = Y_2$ and $(\mu_1 \odot \mu_2)^*((q_1, q_2), x_2, (p_1, p_2), y_2) = \mu_1^*(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$, $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, \forall x_2 \in X_2$ and $\forall y_2 \in Y_2$ and $\omega_x : Q_2 \times X_2 \to X_1$ and $\omega_y : Q_2 \times Y_2 \to Y_1$.

5) the wreath product of fmms of M_1 and M_2, if $Q = Q_1 \times Q_2, X = X_1^{Q_2} \times X_2, Y = Y_1^{Q_2} \times Y_2$ and $(\mu_1 \odot \mu_2)^*((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)) = \mu_1^*(q_1, f(q_2), p_1, g(q_2)) \wedge \mu_2^*(q_2, x_2, p_2, y_2), \forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, \forall x_2 \in X_2$ and $\forall y_2 \in Y_2$ and $X_1^{Q_2} = \{f : Q_2 \to X_1\}$ and $Y_1^{Q_2} = \{g : Q_2 \to Y_1\}$.

Remark 3.1 i) The restricted direct product of M_1 and M_2 is a special case of their cascade product, $X_1 = X_2, Y_1 = Y_2$ and $\omega_x : Q_2 \times X_2 \to X_1, \omega_y : Q_2 \times Y_2 \to Y_1$ both are projection functions.

ii) The cascade product of M_1 and M_2 is a special case of their wreath product, when $X_1^{Q_2} = \{\omega_x\}$ and $Y_1^{Q_2} = \{\omega_y\}$.

Theorem 3.2 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$ and $Y_1 \cap Y_2 = \phi$. Let $M_1 \odot M_2 = (Q_1 \times Q_2, X_1 \cup X_2, Y_1 \cup Y_2, \mu_1 \odot \mu_2)$ be the cartesian product of M_1 and M_2. Then $\forall x \in X_1^* \cup X_2^*, x \neq \lambda, \forall y \in Y_1^* \cup Y_2^*, y \neq \lambda$
$(\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) =$

$$= \begin{cases} \mu_1^*(q_1, x, p_1, y) & \text{if } x \in X_1^*, y \in Y_1^* \text{ and } q_2 = p_2 \\ \mu_2^*(q_2, x, p_2, y) & \text{if } x \in X_2^*, y \in Y_2^* \text{ and } q_1 = p_1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$

Proof. Let $x \in X_1^* \cup X_2^*, x \neq \lambda, y \in Y_1^* \cup Y_2^*, y \neq \lambda$ and let $\mid x \mid = \mid y \mid = n$. Suppose that $x \in X_1^*$ and $y \in Y_1^*$. Clearly the result is true if $n = 1$. Suppose the result is true $\forall u \in X_1^*, \mid u \mid = n - 1, n > 1$ and $\forall v \in Y_1^*, \mid v \mid = n - 1, n > 1$. Let $x = au$ where $a \in X_1$ and $u \in X_1^*$ and $y = bv$ where $b \in Y_1$ and $v \in Y_1^*$. Now, $(\mu_1 \odot \mu_2)^*((p_1, p_2), au, (q_1, q_2), bv) =$

$$\begin{aligned} &= \vee \{(\mu_1 \odot \mu_2)((p_1, p_2), a, (r_1, r_2), b) \\ &\quad \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), u, (q_1, q_2), v) \mid \\ &\quad (r_1, r_2) \in Q_1 \times Q_2\} \\ &= \vee \{\mu_1(p_1, a, r_1, b) \wedge (\mu_1 \odot \mu_2)^*((r_1, p_2), u, (q_1, q_2), v) \\ &\quad \mid r_1 \in Q_1\} \\ &= \begin{cases} \vee\{\mu_1(p_1, a, r_1, b) \wedge \mu_1^*(r_1, u, q_1, v) \mid r_1 \in Q_1\} & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_1^*(p_1, au, q_1, bv) & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3)$$

The result is now follows by induction. Similarly, if $x \in X_2^*$ and $y \in Y_2^*$ one can prove the other case.

Theorem 3.3 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$ and $Y_1 \cap Y_2 = \phi$. Let $M_1 \odot M_2 = (Q_1 \times Q_2, X_1 \cup X_2, Y_1 \cup Y_2, \mu_1 \odot \mu_2)$ be the cartesian product of M_1 and M_2. Then $\forall x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$
$(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2) =$
$\mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$
$= (\mu_1 \odot \mu_2)^*((q_1, q_2), x_2x_1, (p_1, p_2), y_2y_1)$

Proof. Let $x_1 \in X_1^*, x_2 \in X_2^*, y_1 \in Y_1^*, y_2 \in Y_2^*$ and $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$.
case(i) If $x_1 = x_2 = y_1 = y_2 = \lambda$ then $x_1x_2 = y_1y_2 = \lambda$.
case(ii) Suppose, $(q_1, q_2) = (p_1, p_2)$, then $q_1 = p_1$ or $q_2 = p_2$. Hence, $(\mu_1 \odot \mu_2)((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2) = 1 = 1 \wedge 1 = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$.
suppose,$(q_1, q_2) \neq (p_1, p_2)$, then either $q_1 \neq p_1$ and $q_2 \neq p_2$. Thus, $\mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2) = 0$. Hence $(\mu_1 \odot \mu_2)((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2) = 0 = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$.
case(iii) If $x_1 = \lambda, y_1 = \lambda$ and $x_2 \neq \lambda, y_2 \neq \lambda$ or $x_1 \neq \lambda, y_1 \neq \lambda$ and $x_2 = \lambda, y_2 = \lambda$.Then by theorem 3.2 result holds.
case(iv) Suppose, $x_1 \neq \lambda, y_1 \neq \lambda$ and $x_2 \neq \lambda, y_2 \neq \lambda$. Now $(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2) =$

$$\begin{aligned}
&= \bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1, (r_1, r_2), y_1) \\
&\quad \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_2, (p_1, p_2), y_2) \mid \\
&\quad (r_1, r_2) \in Q_1 \times Q_2\} \\
&= \bigvee\{\bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1, (r_1, r_2), y_1) \\
&\quad \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_2, (p_1, p_2), y_2) \mid r_2 \in Q_2\} \mid \\
&\quad r_1 \in Q_1\} \\
&= \bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1, (r_1, q_2), y_1) \\
&\quad \wedge (\mu_1 \odot \mu_2)^*((r_1, q_2), x_2, (p_1, p_2), y_2) \mid r_1 \in Q_1\} \\
&= \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)
\end{aligned} \tag{4}$$

Similarly,$(\mu_1 \odot \mu_2)^*((q_1, q_2), x_2x_1, (p_1, p_2), y_2y_1) = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2)$.
Theorem 3.4 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$ and $Y_1 \cap Y_2 = \phi$. Let $M_1 \odot M_2 = (Q_1 \times Q_2, X_1 \cup X_2, Y_1 \cup Y_2, \mu_1 \odot \mu_2)$
be the cartesian product of M_1 and M_2. Then $\forall x \in (X_1 \cup X_2)^* \exists x_1 \in X_1^*, x_2 \in X_2^*$ and $\forall y \in (Y_1 \cup Y_2)^* \exists y_1 \in Y_1^*, y_2 \in Y_2^*$ such that $(\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) = (\mu_1 \odot \mu_2)^*((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2)$
$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$.
Proof. Let $x \in (X_1 \cup X_2)^*, y \in (Y_1 \cup Y_2)^*$ and $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$. If $x = \lambda = y$, then we can choose $x_1 = x_2 = y_1 = y_2 = \lambda$. In this case the result is trivially true. Suppose $x \neq \lambda, y \neq \lambda$. If $x \in X_1^*$ or $x \in X_2^*, y \in Y_1^*$ or $y \in Y_2^*$, then again the result is trivially true. Suppose $x \notin X_1^* x \notin X_2^*$ and $y \notin Y_1^*$ or $y \notin Y_2^*$
case(I) If $x = x_1x_2, x_1 \in X_1^* - \lambda, x_2 \in X_2^* - \lambda$ and $y = y_1y_2, y_1 \in Y_1^* - \lambda, y_2 \in Y_2^* - \lambda$. Then result follows by Theorem (3.3)
case(II) Suppose $x = x_{11}x_{21}x_{12}$ where $x_{11}, x_{12} \in X_1^*, x_{21} \in X_2^*$ and $y = y_{11}y_{21}y_{12}$, where $y_{11}, y_{12} \in Y_1^*, y_{21} \in Y_2^*$, $x_{1i}, x_{21}, y_{1i}, y_{21}$ are non-empty strings, $i = 1, 2$. Let $x_1 = x_{11}x_{12} \in X_1^*$ and $x_2 = x_{21}$ and $y_1 = y_{11}y_{12} \in Y_1^*$ and $y_2 = y_{21}$. Then $(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}x_{21}x_{12}, (p_1, p_2), y_{11}y_{21}y_{12}) = \bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}, (r_1, r_2), y_{11}) \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_{21}x_{12}, (p_1, p_2), y_{21}y_{12}) | (r_1, r_2) \in Q_1 \times Q_2\}$ $= \bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}, (r_1, r_2), y_{11}) \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_{12}x_{21}, (p_1, p_2), y_{12}y_{21}) | (r_1, r_2) \in Q_1 \times Q_2\}$ $= (\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}\, x_{12}x_{21}, (p_1, p_2), y_{11}y_{12}y_{21})$
case(III) Suppose $x = x_{21}x_{11}x_{22}$ where $x_{21}, x_{22} \in X_2^*, x_{11} \in X_1^*$ and $y = y_{21}y_{11}y_{22}$ where $y_{21}, y_{22} \in Y_2^*, y_{11} \in Y_1^*$, $x_{21}, x_{2i}, y_{21}, y_{2i}$ are non-empty strings $i = 1, 2$. Let $x_2 = x_{21}x_{22} \in X_2^*$ and $x_1 = x_{11}$ and $y_2 = y_{21}y_{22} \in Y_2^*$ and $y_1 = y_{11}$. The proof of this case is similar to case(II)
case(IV) Suppose $x = x_{11}x_{21}x_{12}x_{22}$, where $x_{11}, x_{12} \in X_1^*, x_{21}, x_{22} \in X_2^*$ and $y = y_{11}y_{21}y_{12}y_{22}$, where $y_{11}, y_{12} \in Y_1^*, y_{21}, y_{22} \in Y_2^*$, $x_{1i}, x_{2i}, y_{1i}, y_{2i}$ are non-empty strings, $i = 1, 2$. Let $x_1 = x_{11}x_{12} \in X_1^*$ $x_2 = x_{21}x_{22} \in X_2^*$ and $y_1 = y_{11}y_{12} \in Y_1^*$ $y_2 = y_{21}y_{22} \in Y_2^*$
Then $(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}x_{21}x_{12}x_{22}, (p_1, p_2), y_{11}y_{21}y_{12}y_{22}) = \bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}, (r_1, r_2), y_{11}) \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_{21}x_{12}x_{22}, (p_1, p_2), y_{21}y_{12}y_{22}) | (r_1, r_2) \in Q_1 \times Q_2\}$= $\bigvee\{(\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}, (r_1, r_2), y_{11}) \wedge (\mu_1 \odot \mu_2)^*((r_1, r_2), x_{12}x_{21}x_{22}, (p_1, p_2), y_{12}y_{21}y_{22}) | (r_1, r_2) \in Q_1 \times Q_2\}$ (by case (III)) $= (\mu_1 \odot \mu_2)^*((q_1, q_2), x_{11}x_{12}x_{21}x_{22}, (p_1, p_2), y_{11}y_{12}y_{21}y_{22})$.
case(V)Suppose $x = x_{21}x_{11}x_{22}x_{12}$, where $x_{11}, x_{12} \in X_1^*, x_{21}, x_{22} \in X_2^*$ and $y = y_{21}y_{11}y_{22}y_{12}$, where $y_{11}, y_{12} \in Y_1^*, y_{21}, y_{22} \in Y_2^*$, $x_{1i}, x_{2i}, y_{1i}, y_{2i}$ are non-empty strings, $i = 1, 2$. Let $x_1 = x_{11}x_{12} \in X_1^*$ $x_2 = x_{21}x_{22} \in X_2^*$ and $y_1 = y_{11}y_{12} \in Y_1^*$ $y_2 = y_{21}y_{22} \in Y_2^*$. The proof of this case is similar to case(IV). The theorem now follows by induction.
Theorem 3.5 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$. Then fmms $M_1 \odot M_2$ is cyclic if and only if M_1 and M_2 are cyclic, where $\odot$ is cartesian product and full direct product.
Proof. **1)** When $\odot$ is cartesian product. Suppose M_1 and M_2 are cyclic, say $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then $\exists x_1 \in X_1^*, x_2 \in X_2^*, y_1 \in Y_1^*, y_2 \in Y_2^*$ such that $\mu_1^*(q_1, x_1, p_1, y_1) > 0$ and $\mu_2^*(q_2, x_2, p_2, y_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2) = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2) > 0$. That is, $(p_1, p_2) \in S(q_1, q_2)$. Thus, $Q_1 \times Q_2 = S((q_1, q_2))$.Hence $M_1 \odot M_2$ is cyclic.
Conversely, Suppose $M_1 \odot M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then $\exists x \in (X_1 \cup X_2)^*$ and $\exists y \in (Y_1 \cup Y_2)^*$ such that $(\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) > 0$. Then by theorem 3.3 $\exists x_1 \in X_1^*, x_2 \in X_2^*$ and $\exists y_1 \in Y_1^*, y_2 \in Y_2^*$ such that $\mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2) = (\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) > 0$. That is $\exists x_1 \in X_1^*, x_2 \in X_2^*$ and $\exists y_1 \in Y_1^*, y_2 \in Y_2^*$ such that $\mu_1^*(q_1, x_1, p_1, y_1) > 0$ and $\mu_2^*(q_2, x_2, p_2, y_2) > 0$. That is $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. That is $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$. Thus M_1 and M_2 are cyclic.
2)When $\odot$ is full direct product. Suppose M_1 and M_2 are cyclic, say $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then $\mu_1^*(q_1, x_1, p_1, y_1) > 0$ and $\mu_2^*(q_2, x_2, p_2, y_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)) = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2) > 0$. That is, $(p_1, p_2) \in S((q_1, q_2))$. Thus, $Q_1 \odot Q_2 = S(q_1, q_2)$.Hence $M_1 \odot M_2$ is cyclic.
Conversely, Suppose $M_1 \odot M_2$ is cyclic. Let $Q_1 \odot Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \odot Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then $(\mu_1 \odot \mu_2)^*((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)) = \mu_1^*(q_1, x_1, p_1, y_1) \wedge \mu_2^*(q_2, x_2, p_2, y_2) > 0$. This implies $\mu_1^*(q_1, x_1, p_1, y_1) > 0$ and $\mu_2^*(q_2, x_2, p_2, y_2) > 0$. Therefore, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$.That is $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence, M_1 and M_2 are cyclic.
Theorem 3.6 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$ If fmms $M_1 \odot M_2$ is cyclic then M_1 and M_2 are cyclic, where $\odot$ is restricted direct product, cascade product and wreath product.
Proof. **1)** When $\odot$ is restricted direct product. Suppose $M_1 \odot M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then $(\mu_1 \odot \mu_2)^*((q_1, q_2), x, (p_1, p_2), y) = \mu_1^*(q_1, x, p_1, y) \odot \mu_2^*(q_2, x, p_2, y) > 0$. This implies $\mu_1^*(q_1, x, p_1, y) > 0$ and $\mu_2^*(q_2, x, p_2, y) > 0$. Therefore, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. That is $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence, M_1 and M_2 are cyclic.
2) When $\odot$ is cascade product. Suppose $M_1 \odot M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in$

Q_1 and $p_2 \in Q_2$. Then $(\mu_1 \odot \mu_2)^*((q_1,q_2),x_2,(p_1,p_2),y_2) = \mu_1^*(q_1,\omega_x(q_2,x_2),p_1,\omega_y\ (q_2,y_2)) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. This implies $\mu_1^*(q_1,\omega_x(q_2,x_2),p_1,\omega_y\ (q_2,y_2)) > 0$ and $\mu_2^*(q_2,x_2,p_2,y_2) > 0$. Therefore, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$.That is $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence, M_1 and M_2 are cyclic.
3) When $\odot$ is wreath product. Suppose $M_1 \odot M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1,q_2))$, for some $(q_1,q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then $(\mu_1 \odot \mu_2)^*((q_1,q_2),(f,x_2),(p_1,p_2),(g\ ,y_2)) = \mu_1^*(q_1,f(q_2),p_1,g(q_2)) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. This implies $\mu_1^*(q_1,f(q_2),p_1,g(q_2)) > 0$ and $\mu_2^*(q_2,x_2,p_2,y_2) > 0$. Therefore, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$.That is $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence, M_1 and M_2 are cyclic.
The converse of the above theorem is true when individual fmms are strongly connected.

Theorem 3.7 Let $M_i = (Q_i,X_i,Y_i,\mu_i)$ be a strongly connected fmms, $i = 1,2$. Then $M_1 \odot M_2$ is cyclic , where $\odot$ is restricted direct product, cascade product and wreath product.

Proof. **1)** When $\odot$ is restricted direct product. By theorem (3.1) M_1 and M_2 are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1,p_2) \in Q_1 \times Q_2$. Then $\mu_1^*(q_1,x,p_1,y) > 0$ and $\mu_2^*(q_2,x,p_2,y) > 0$. Thus $(\mu_1 \odot \mu_2)^*((q_1,q_2),x,(p_1,p_2),y) = \mu_1^*(q_1,x,p_1,y) \wedge \mu_2^*(q_2,x,p_2,y) > 0$. That is, $(p_1,p_2) \in S((q_1,q_2))$. Thus, $Q_1 \times Q_2 = S(q_1,q_2)$.Hence $M_1 \odot M_2$ is cyclic.
2) When $\odot$ is cascade product. Since M_1 and M_2 are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1,p_2) \in Q_1 \times Q_2$. Then $\mu_1^*(q_1,\omega_x(q_2,x_2),p_1,\omega_y\ (q_2,y_2)) > 0$ and $\mu_2^*(q_2,x_2,p_2,y_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*((q_1,q_2),x_2,(p_1,p_2),y_2) = \mu_1^*(q_1,\omega_x\ (q_2,x_2),p_1,\ \omega_y(q_2,y_2)) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. That is, $(p_1,p_2) \in S((q_1,q_2))$. Thus, $Q_1 \times Q_2 = S(q_1,q_2)$.Hence $M_1 \odot M_2$ is cyclic.
3) When $\odot$ is wreath product. Since M_1 and M_2 are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1,p_2) \in Q_1 \times Q_2$. Then $\mu_1^*(q_1,f(q_2),p_1,g(q_2)) > 0$ and $\mu_2^*(q_2,x_2,p_2,y_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*((q_1,q_2),(f,x_2),(p_1,p_2),(g,y_2)) = \mu_1^*(q_1,f(q_2),p_1,g\ (q_2)) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. That is, $(p_1,p_2) \in S((q_1,q_2))$. Thus, $Q_1 \times Q_2 = S(q_1,q_2)$. Hence $M_1 \odot M_2$ is cyclic.

Theorem 3.8 Let $M_i = (Q_i,X_i,Y_i,\mu_i)$ be a fmms, $i = 1,2$. Then fmms $M_1 \odot M_2$ is retrievable if and only if M_1 and M_2 are retrievable, where $\odot$ is cartesian product and full direct product.

Proof. **1)** When $\odot$ is cartesian product.Suppose M_1 and M_2 are retrievable. Let $(q_1,q_2),(p_1,\ p_2) \in Q_1 \times Q_2$ and $x \in (X_1 \cup X_2)^*, y \in (Y_1 \cup Y_2)^*$ be such that$(\mu_1 \odot \mu_2)^*((q_1,q_2),x,(p_1,p_2),y) > 0$. Let $x_1^* = x_1x_2$ be the standard form of x, $x_1 \in X_1$ and $x_2 \in X_2$ and $y_1^* = y_1y_2$ be the standard form of y, $y_1 \in Y_1$ and $y_2 \in Y_2$. Then $(\mu_1 \odot \mu_2)^*((q_1,q_2),x,(p_1,p_2),y) = (\mu_1 \odot \mu_2)^*((q_1,q_2),x_1x_2,(p_1,p_2),y_1y_2) = \mu_1^*(q_1,x_1,p_1,y_1) \wedge \mu_2^*(q_2,x_2,p_2,y_2)$. Thus $\mu_1^*(q_1,x_1,p_1,y_1) > 0$ and $\mu_2^*(q_2,x_2,p_2,y_2) > 0$. Since M_1 and M_2 are retrievable,$\exists u_1 \in X_1^*, u_2 \in X_2^*, v_1 \in Y_1^*, v_2 \in Y_2^*$ such that $\mu_1^*(p_1,u_1,q_1,v_1) > 0$ and $\mu_2^*(p_2,u_2,q_2,v_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*((p_1,p_2),u_1u_2,(q_1,q_2),v_1v_2) > 0$. That is $(\mu_1 \odot \mu_2)^*((p_1,p_2),u,(q_1,q_2),v) > 0$. Hence, $M_1 \odot M_2$ is retrievable.
Conversely, suppose $M_1 \odot M_2$ is retrievable. Let $q_1,p_1 \in Q_1$, $x \in X_1^*$ and $y \in Y_1^*$ be such that $\mu_1^*(q_1,x_1,p_1,y_1) > 0$. Then $\forall q_2 \in Q_2$, $(\mu_1 \odot \mu_2)^*((q_1,q_2),x_1,(p_1,q_2),y_1) > 0$ Thus $\exists u \in (X_1 \cup X_2)^*$ and $\exists v \in (Y_1 \cup Y_2)^*$ such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),u,(p_1,q_2),v) > 0$. Let $u = u_1u_2$ be the standard form of u and v respectively where $u_1 \in X_1^*$, $u_2 \in X_2^*$ and $v_1 \in Y_1^*$, $v_2 \in Y_2^*$. Then $0 < (\mu_1 \odot \mu_2)^*((q_1,q_2),u_1u_2,(p_1,q_2),v_1v_2) = \mu_1(q_1,u_1,p_1,v_1) \wedge \mu_2(q_2,u_2,q_2,v_2)$. Thus $\mu_1(q_1,u_1,p_1,v_1) > 0$. Hence M_1 is retrievable. Similarly M_2 is retrievable.
2)When $\odot$ is full direct product. Suppose M_1 and M_2 are retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$ and $(x_1,x_2) \in (X_1 \times X_2)^*$,$(y_1,y_2) \in (Y_1 \times Y_2)^*$ be such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),(x_1,x_2),\ (p_1,p_2),(y_1,y_2)) = \mu_1^*(q_1,x_1,p_1,y_1) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. Since, M_1 and M_2 are retrievable. $\exists u_1 \in X_1^*, v_1 \in Y_1^*$ and $\exists u_2 \in X_2^*, v_2 \in Y_2^*$, such that $\mu_1^*(p_1,u_1,q_1,v_1) > 0$ and $\mu_2^*(p_2,u_2,q_2,v_2) > 0$. Now $\mu_1^*(p_1,u_1,q_1,v_1) \wedge \mu_2^*(p_2,u_2,q_2,v_2) = (\mu_1 \times \mu_2)^*((p_1,p_2),(u_1,u_2),(q_1,q_2),(v_1,v_2)) > 0$. Therefore,$M_1 \odot M_2$ is retrievable.
Conversely, suppose $M_1 \odot M_2$ is retrievable. Let $(q_1,q_2) \in Q_1 \times Q_2,(x_1,x_2) \in (X_1 \times X_2)^*$ and $(y_1,y_2) \in (Y_1 \times Y_2)^* \exists(p_1,p_2) \in Q_1 \times Q_2$ such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),(x_1,x_2),(p_1,p_2),\ (y_1,y_2)) > 0$ then $\exists(u_1,u_2) \in (X_1 \times X_2)^*,(v_1,v_2) \in (Y_1 \times Y_2)^*$ such that $(\mu_1 \odot \mu_2)^*((p_1,p_2),\ (u_1,u_2),(q_1,q_2),(v_1,v_2)) = \mu_1^*(p_1,u_1,q_1,v_1) \wedge \mu_2^*(p_2,u_2,q_2,v_2) > 0$. Hence, M_1 and M_2 are retrievable.

Theorem 3.9 Let $M_i = (Q_i,X_i,Y_i,\mu_i)$ be a fmms, $i = 1,2$ If fmms $M_1 \odot M_2$ is retrievable then M_1 and M_2 are retrievable, where $\odot$ is restricted direct product, cascade product and wreath product.

Proof. **1)** When $\odot$ is restricted direct product. Suppose $M_1 \odot M_2$ is retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, x \in X$ and $y \in Y$ such that $(\mu_1 \times \mu_2)^*((q_1,q_2),x,(p_1,p_2),y) > 0$ then $\exists u \in X^*, v \in Y^*$ such that $(\mu_1 \times \mu_2)^*((p_1,p_2),u,(q_1,q_2),v) = \mu_1^*(p_1,u,q_1,v) \wedge \mu_2^*(p_2,u,q_2,v) > 0$. Hence, M_1 and M_2 are retrievable.
2) When $\odot$ is cascade product. Suppose $M_1 \odot M_2$ is retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, x_2 \in X_2$ and $y_2 \in Y_2$ such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),x_2,(p_1,p_2),y_2) > 0$ then $\exists u_2 \in X_2^*, v_2 \in Y_2^*$ such that $(\mu_1 \odot \mu_2)^*((p_1,p_2),u_2,(q_1,q_2),v_2) = \mu_1^*(p_1,\omega_u(p_2,u_2),q_1,\omega_v(p_2,\ v_2)) \wedge \mu_2^*(p_2,u_2,\ q_2,v_2) > 0$. Hence, M_1 and M_2 are retrievable.
3) When $\odot$ is wreath product. Suppose $M_1 \odot M_2$ is retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2, x_2 \in X_2^*$ and $y_2 \in Y_2^*$ such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),(f,x_2),(p_1,p_2),(g,y_2)) > 0$ then $\exists u_2 \in X_2^*, v_2 \in Y_2^*$ such that $(\mu_1 \odot \mu_2)^*((p_1,p_2),(f,u_2),(q_1,q_2),(g,v_2)) = \mu_1^*(p_1,f(p_2),q_1,g(p_2)) \wedge \mu_2^*(p_2,u_2,q_2,v_2) > 0$. Hence, M_1 and M_2 are retrievable.
The converse of the above theorem is true when individual fmms are strongly connected.

Theorem 3.10 Let $M_i = (Q_i,X_i,Y_i,\mu_i)$ be a strongly connected fmms, $i = 1,2$. Then $M_1 \odot M_2$ is retrievable , where $\odot$ is restricted direct product, cascade product and wreath product.

Proof. **1)** When $\odot$ is restricted direct product. By theorem (3.1) M_1 and M_2 are retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$ and $x \in X^*$,$y \in Y^*$ be such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),x,(p_1,p_2),y) = \mu_1^*(q_1,x,p_1,y) \wedge \mu_2^*(q_2,x,p_2,y) > 0$. Since, M_1 and M_2 are retrievable. $\exists u \in X^*, v \in Y^*$, such that $\mu_1^*(p_1,u,q_1,v) > 0$ and $\mu_2^*(p_2,u,q_2,v) > 0$. Now $\mu_1^*(p_1,u,q_1,v) \wedge \mu_2^*(p_2,u,q_2,v) = (\mu_1 \odot \mu_2)^*((p_1,p_2),u,(q_1,q_2),v) > 0$. Therefore,$M_1 \odot M_2$ is retrievable.
2) When $\odot$ is cascade product. Suppose M_1 and M_2 are retrievable. Let $(q_1,q_2),(p_1,p_2) \in Q_1 \times Q_2$ and $x_2 \in X_2^*$,$y_2 \in Y_2^*$ be such that $(\mu_1 \odot \mu_2)^*((q_1,q_2),x_2,(p_1,p_2),y_2) = \mu_1^*(q_1,\omega_x(q_2,x_2),p_1,\omega_y(q_2,\ \ y_2)) \wedge \mu_2^*(q_2,x_2,p_2,y_2) > 0$. Since, M_1 and M_2 are retrievable. $\exists u_2 \in X_2^*, v_2 \in Y_2^*$, such that $\mu_1^*(p_1,\omega_u(q_2,u_2),q_1,\omega_v(q_2,v_2)) > 0$ and $\mu_2^*(p_2,u_2,q_2,v_2) > 0$. Now $\mu_1^*(p_1,\odot_u(q_2,u_2),q_1,\odot_v(q_2,v_2)) \wedge \mu_2^*(p_2,u_2,q_2,v_2) = (\mu_1 \odot \mu_2)^*((p_1,p_2),u_2,(q_1,q_2),v_2) > 0$. Therefore,$M_1 \odot M_2$

is retrievable.
3) When $\odot$ is wreath product. Suppose M_1 and M_2 are retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $x_2 \in X_2^*$,$y_2 \in Y_2^*$ be such that $(\mu_1 \odot \mu_2)^*((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)) = \mu_1^*(p_1, f(p_2), q_1, g(p_2)) \wedge \mu_2^*(q_2, x_2, p_2, y_2) > 0$. Since, M_1 and M_2 are retrievable. $\exists u_2 \in X_2^*, v_2 \in Y_2^*$, such that $\mu_1^*(p_1, f(p_2), q_1, g(p_2)) > 0$ and $\mu_2^*(p_2, u_2, q_2, v_2) > 0$. Now $\mu_1^*(p_1, f(p_2), q_1, g(p_2)) \wedge \mu_2^*(p_2, u_2, q_2, v_2) = (\mu_1 \odot \mu_2)^*((p_1, p_2), (g, u_2), (q_1, q_2), (g, v_2)) > 0$. Therefore,$M_1 \odot M_2$ is retrievable.
Remark 3.2 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, then by theorem (2.7),(3.8) and (3.9)

(1) Fmm $M_1 \odot M_2$ is union of strongly connected submachines (exchange property) if and only if M_1 and M_2 are union of strongly connected submachines (exchange property), where $\odot$ is cartesian product and full direct product.
(2) Fmm $M_1 \odot M_2$ is union of strongly connected submachines(exchange property) then M_1 and M_2 are union of strongly connected submachines (exchange property), where $\odot$ is restricted direct product, cascade product and wreath product.

Remark 3.3 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a strongly connected fmms, then by theorem (2.7),(3.5) and (3.10) $M_1 \odot M_2$ is union of strongly connected submachines (exchange property), where $\odot$ is restricted direct product, cascade product and wreath product.
Theorem 3.11 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$. Then fmm $M_1 \odot M_2$ is connected if and only if M_1 and M_2 are connected, where $\odot$ is cartesian product and full direct product.
Proof. **1)** When $\odot$ is cartesian product.
Suppose M_1 and M_2 are connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists q_{10}, q_{11}, ..., q_{1n} \in Q_1$, $q_1 = q_{10}, p_1 = q_{1n}$ and $\exists a_{11}, a_{12}, ..., a_{1n} \in X_1$ and $\exists b_{11}, b_{12}, ..., b_{1n} \in Y_1$ $\forall i = 1, 2, ..., n$ either $\mu_1(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ or $\mu_1(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0$ and $\exists q_{20}, q_{21}, ..., q_{2m} \in Q_2$, $q_2 = q_{20}, p_2 = q_{2m}$ and $\exists a_{21}, a_{22}, ..., a_{2m} \in X_2$ and $\exists b_{21}, b_{22}, ..., b_{2m} \in Y_2$ $\forall i = 1, 2, ..., m$ either $\mu_2(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ or $\mu_2(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0$. Consider the sequence of states $(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{20}), ..., (q_{1n}, q_{20}), (q_{1n}, q_{21}), ...(q_{1n}, q_{2m}) = (p_1, p_2) \in Q_1 \times Q_2$ and the sequence $a_{11}, a_{12}, ..., a_{1n}, a_{21}, a_{22}, ..., a_{2m} \in X_1 \cup X_2$ and $b_{11}, b_{12}, ..., b_{1n}, b_{21}, b_{22}, ..., b_{2m} \in Y_1 \cup Y_2$.$\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, q_{20}), a_{1i}, (q_{1i}, q_{20}), b_{1i})) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, q_{20}), a_{1i}, (q_{1i-1}, q_{20}), b_{1i})) > 0$ and $\forall j = 1, 2, ..., m$ either $\mu_1 \odot \mu_2((q_{1n}, q_{2j-1}), a_{2j}, (q_{1n}, q_{2j}), b_{2j})) > 0$ or $\mu_1 \odot \mu_2((q_{1n}, q_{2j}), a_{2j}, (q_{1n}, q_{2j-1}), b_{2j})) > 0$. Hence (q_1, q_2) and (p_1, p_2) are connected. i.e. $M_1 \odot M_2$ is connected
Conversely Suppose that $M_1 \odot M_2$ is connected. Let $q_1, p_1 \in Q_1$ and let $r_2 \in Q_2$. If $p_1 = q_1$ then p_1 and q_1 are connected. Suppose, $p_1 \neq q_1$ Then $\exists$ $(q_1, r_2) = (q_{10}, r_{20}), (q_{11}, r_{21}), ..., (q_{1n}, r_{2n}) = (p_1, r_2) \in Q_1 \times Q_2$ and $a_1, a_2, ..., a_n \in X_1 \cup X_2$ such that $\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, r_{2i-1}), a_i, (q_{1i}, r_{2i}), b_i)) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, r_{2i}), a_i, (q_{1i-1}, r_{2i-1}), b_i)) > 0$ Clearly, if $q_{1i-1} \neq q_{1i}$ then $r_{2i-1} = r_{2i}$ and if $r_{2i-1} \neq r_{2i}$ then $q_{1i-1} = q_{1i}$ $\forall i = 1, 2, ..., n$ Let $\{q_1 = q'_{11}, q'_{12}, q'_{13}, ..., q'_{1k} = p_1\}$ be the set of all distinct $q'_{1i} \in \{q_{10}, q_{11}, ..., q_{1n}\}$ and let $a'_1, a'_2, ..., a'_k \in X_1$ and $b'_1, b'_2, ..., b'_k \in Y_1$be the corresponding a_i's and b_i's respectively and $\forall j = 1, 2, ..., k$ either $\mu_1(q'_{1j-1}, a'_j, q'_{1j}, b'_j) > 0$ or $\mu_1(q'_{1j}, a'_j, q'_{1j-1}, b'_j) > 0$ Thus p_1 and q_1 are connected and hence M_1 is connected. Similarly we can show that M_2 is connected.
2) When $\odot$ is full direct product.
case(i) Suppose M_1 and M_2 are connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists q_{10}, q_{11}, ..., q_{1n} \in Q_1$, $q_1 = q_{10}, p_1 = q_{1n}$ and $\exists a_{11}, a_{12}, ..., a_{1n} \in X_1$ and $\exists b_{11}, b_{12}, ..., b_{1n} \in Y_1$ $\forall i = 1, 2, ..., n$ $\mu_1(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ and $\exists q_{20}, q_{21}, ..., q_{2m} \in Q_2$, $q_2 = q_{20}, p_2 = q_{2m}$ and $\exists a_{21}, a_{22}, ..., a_{2m} \in X_2$ and $\exists b_{21}, b_{22}, ..., b_{2m} \in Y_2$. $\forall i = 1, 2, ..., m$ $\mu_2(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ Without loss of generality $m \leq n$. Consider the sequence of states $(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{21}), ..., (q_{1m}, q_{2m}), (q_{1m+1}, q_{2m}), ..., (q_{1n}, q_{2m}) = (p_1, p_2)$ and a sequence $(a_{11}, a_{21}), ..., (a_{1m}, a_{2m}), (a_{1m+1}, a_{2m+1}), ..., (a_{1n}, a_{2n})$ where $a_{2k} = \lambda, \forall k = m+1, ..., n$. and a sequence $(b_{11}, b_{21}), ..., (b_{1m}, b_{2m}), (b_{1m+1}, b_{2m+1}), ..., (b_{1n}, b_{2n})$ where $b_{2k} = \lambda, \forall k = m+1, ..., n$. Then $\forall i = 1, 2, ..., n$,$\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})) > 0$, where $q_{2i} = q_{2m}, \forall i = m+1, ..., n$. Hence (q_1, q_2) and $(p_1, p_2)$are connected. i.e. $M_1 \odot M_2$ is connected.
case(ii) Suppose M_1 and M_2 are connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists q_{10}, q_{11}, ..., q_{1n} \in Q_1$, $q_1 = q_{10}, p_1 = q_{1n}$ and $\exists a_{11}, a_{12}, ..., a_{1n} \in X_1$ and $\exists b_{11}, b_{12}, ..., b_{1n} \in Y_1$ $\forall i = 1, 2, ..., n$ $\mu_1(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0$ and $\exists q_{20}, q_{21}, ..., q_{2m} \in Q_2$, $q_2 = q_{20}, p_2 = q_{2m}$ and $\exists a_{21}, a_{22}, ..., a_{2m} \in X_2$ and $\exists b_{21}, b_{22}, ..., b_{2m} \in Y_2$. $\forall i = 1, 2, ..., m$ $\mu_2(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0$ Without loss of generality $m \leq n$. Consider the sequence of states $(p_1, p_2) = (q_{1n}, q_{2m}), (q_{1n-1}, q_{2m}), ..., (q_{1m}, q_{2m}), (q_{1m-1}, q_{2m-1}), ..., (q_{11}, q_{21}), (q_{10}, q_{20}) = (q_1, q_2)$ and a sequence $(a_{1n}, a_{2n}), ..., (a_{1m}, a_{2m}), (a_{1m-1}, a_{2m-1}), ..., (a_{10}, a_{20})$ where $a_{2k} = \lambda, \forall k = m+1, ..., n$. and a sequence $(b_{1n}, b_{2n}), ..., (b_{1m}, b_{2m}), (b_{1m-1}, b_{2m-1}), ..., (b_{10}, b_{20})$ where $b_{2k} = \lambda, \forall k = m+1, ..., n$.. Then $\forall i = n, n-1, ..., 1$,$\mu_1 \odot \mu_2((q_{1i}, q_{2i}), a_{1i}, (q_{1i}, q_{2i}), b_{1i})) > 0$, where $q_{2i} = q_{2m}, \forall i = m+1, ..., n$.Hence (q_1, q_2) and $(p_1, p_2)$are connected. i.e. $M_1 \odot M_2$ is connected.
case(iii) Suppose M_1 and M_2 are connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists q_{10}, q_{11}, ..., q_{1n} \in Q_1$, $q_1 = q_{10}, p_1 = q_{1n}$ and $\exists a_{11}, a_{12}, ..., a_{1n} \in X_1$ and $\exists b_{11}, b_{12}, ..., b_{1n} \in Y_1$ $\forall i = 1, 2, ..., n$ $\mu_1(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ and $\exists q_{20}, q_{21}, ..., q_{2m} \in Q_2$, $q_2 = q_{2m}, p_2 = q_{20}$ and $\exists a_{21}, a_{22}, ..., a_{2m} \in X_2$ and $\exists b_{21}, b_{22}, ..., b_{2m} \in Y_2$. $\forall i = 1, 2, ..., m$ $\mu_2(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0$ Without loss of generality $m \leq n$. Consider the sequence of states $(q_1, q_2) = (q_{10}, q_{2m}), (q_{11}, q_{2m-1}), ..., (q_{1m}, q_{20}), (q_{1m+1}, q_{20}), ..., (q_{1n}, q_{20}) = (p_1, p_2)$ and a sequence $(a_{11}, a_{2m}), ..., (a_{1m}, a_{21}), (a_{1m+1}, a_{2m+1}), ..., (a_{1n}, a_{2n})$ where $a_{2k} = \lambda, \forall k = m+1, ..., n$ and a sequence $(b_{11}, b_{2m}), ..., (b_{1m}, b_{21}), (b_{1m+1}, b_{2m+1}), ..., (b_{1n}, b_{2n})$ where $b_{2k} = \lambda \forall k = m+1, ..., n$.Hence (q_1, q_2) and $(p_1, p_2)$are connected. i.e. $M_1 \odot M_2$ is connected.
case(iv) Suppose M_1 and M_2 are connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists q_{10}, q_{11}, ..., q_{1n} \in Q_1$, $p_1 = q_{10}, q_1 = q_{1n}$ and $\exists a_{11}, a_{12}, ..., a_{1n} \in X_1$ and $\exists b_{11}, b_{12}, ..., b_{1n} \in Y_1$ $\forall i = 1, 2, ..., n$ $\mu_1(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0$ and $\exists q_{20}, q_{21}, ..., q_{2m} \in Q_2$, $q_2 = q_{20}, p_2 = q_{2m}$ and $\exists a_{21}, a_{22}, ..., a_{2m} \in X_2$ and $\exists b_{21}, b_{22}, ..., b_{2m} \in Y_2$. $\forall i = 1, 2, ..., m$ $\mu_2(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ Without loss of generality $m \leq n$. Consider the sequence of states $(q_1, q_2) = (q_{1n}, q_{20}), (q_{1n-1}, q_{20}), ..., (q_{1m+1}, q_{20})(q_{1m}, q_{20}), (q_{1m-1}, q_{21}), ..., (q_{10}, q_{2m}) = (p_1, p_2)$ and a sequence $(a_{1n}, a_{2n}), ..., (a_{1m}, a_{21}), (a_{1m-1}, a_{22}), ..., (a_{1n}, a_{2m})$ where $a_{2k} = \lambda, \forall k = m+1, ..., n$ and a sequence $(b_{1n}, b_{2n}), ..., (b_{1m}, b_{21}), (b_{1m-1}, b_{22}), ..., (b_{1n}, b_{2m})$ where $b_{2k} = \lambda, \forall k = m+1, ..., n$.Hence (q_1, q_2) and $(p_1, p_2)$are connected. i.e. $M_1 \odot M_2$ is connected.
Conversely Suppose $M_1 \odot M_2$ is connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists$ a sequence of states $\{(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{21}), ..., (q_{1n}, q_{2n}) = (p_1, p_2)\} \in Q_1 \times Q_2$ and the sequence

$\{(a_{11}, a_{21}), (a_{12} \ , a_{22}), ..., (a_{1n}, a_{2n})\} \in X_1 \times X_2$ and $\{(b_{11}, b_{21}), (b_{12}, b_{22}), ..., (b_{1n}, b_{2n})\} \in Y_1 \times Y_2$. $\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, q_{2i}), (a_{1i}, a_{2i}), (q_{1i-1}, q_{2i-1}), (b_{1i}, b_{2i})) > 0$. Without loss of generality, suppose $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})) > 0$. Consider the sequence $\{q_1 = q_{10}, q_{11}, ..., q_{1n} = p_1\}$ and the sequence $\{a_{11}, a_{12}, ..., a_{1n}\} \in X_1$ and $\{b_{11}, b_{12}, ..., b_{1n}\} \in Y_1$ such that $\forall i = 1, ..., n, \mu_1(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$, hence M_1 is connected. Consider the sequence $\{q_2 = q_{20}, q_{21}, ..., q_{2n} = p_2\}$ and the sequence $\{a_{21}, a_{22}, ..., a_{2n}\} \in X_2$ and $\{b_{21}, b_{22}, ..., b_{2n}\} \in Y_2$ such that $\forall i = 1, ..., n,\ \mu_2(q_{2i-1}, a_{2i}, q_{2i},\ b_{2i}) > 0$, hence M_2 is connected.

Theorem 3.12 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms, $i = 1, 2$ If fmm $M_1 \odot M_2$ is connected then M_1 and M_2 are connected, where $\odot$ is restricted direct product,cascade product and wreath product.

Proof. **1)** Suppose $\odot$ is restricted direct product. Suppose $M_1 \odot M_2$ is connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), ..., (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$ and the sequence $\{a_1, a_2, ..., a_n\} \in X$ and $\{b_1, b_2, ..., b_n\} \in Y$ $\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, q_{2i}), a_i, (q_{1i-1}, q_{2i-1}), b_i) > 0$. Without loss of generality, suppose $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i) > 0$. Consider the sequence $\{q_1 = q_{10}, q_{11}, ..., q_{1n} = p_1\}$ and the sequence $\{a_1, a_2, ..., a_n\} \in X$ and $\{b_1, b_2, ..., b_n\} \in Y$ such that $\forall i = 1, ..., n, \mu_1(q_{1i-1},\ a_i, q_{1i}, b_i) > 0$ and consider the sequence $\{q_2 = q_{20}, q_{21}, ..., q_{2n} = p_2\}$ and the sequence $\{a_1, a_2, ..., a_n\} \in X$ and $\{b_1, b_2, ..., b_n\} \in Y$ such that $\forall i = 1, ..., n, \mu_2(q_{2i-1}, a_i, q_{2i}, b_i) > 0$.

2) Suppose $\odot$ is cascade product. Suppose $M_1 \odot M_2$ is connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), ..., (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$ and the sequence $\{x_{21}, x_{22}, ..., x_{2n}\} \in X_2$ and $\{y_{21}, y_{22}, ..., y_{2n}\} \in Y_2$ $\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, q_{2i}), x_{2i}, (q_{1i-1}, q_{2i-1}), y_{2i}) > 0$. Without loss of generality, suppose $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}) > 0$. Consider the sequence $\{q_1 = q_{10}, q_{11}, ..., q_{1n} = p_1\}$ and the sequence $\{\omega_x(q_{21}) = x_{11}(\text{say}), \omega_x(q_{22}) = x_{12}, ..., \omega_x(q_{2n}) = x_{1n}\} \in X_1$ and $\{\omega_y(q_{21}) = y_{11}, \omega_y(q_{22}) = y_{12}, ..., \omega_y(q_{2n}) = y_{1n}\} \in Y_1$ such that $\forall i = 1, ..., n, \mu_1(q_{1i-1},\ x_{1i}, q_{1i}, y_{1i}) > 0$. Now consider the sequence $\{q_2 = q_{20}, q_{21}, ..., q_{2n} = p_2\}$ and the sequence $\{x_{21}, x_{22}, ..., x_{2n}\} \in X_2$ and $\{y_{21}, y_{22}, ..., y_{2n}\} \in Y_2$ such that $\forall i = 1, ..., n, \mu_2(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) > 0$.

3) Suppose $\odot$ is wreath product. Suppose $M_1 \odot M_2$ is connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, Now $\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), ..., (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$ and the sequence $\{(f_1, x_{21}), (f_2, x_{22}), ..., (f_n,\ x_{2n})\} \in X_1^{Q_2} \times X_2$ and $\{(g_1, y_{21}), (g_2, y_{22}), ..., (g_n, y_{2n})\} \in Y_1^{Q_2} \times Y_2$, $\forall i = 1, 2, ..., n$ either $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i, y_{2i})) > 0$ or $\mu_1 \odot \mu_2((q_{1i}, q_{2i}), (f_i, x_{2i}), (q_{1i-1}, q_{2i-1}), (g_i, y_{2i})) > 0$. Without loss of generality, suppose $\mu_1 \odot \mu_2((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i,\ y_{2i})) > 0$. Consider the sequence $\{q_1 = q_{10}, q_{11}, ..., q_{1n} = p_1\}$ and the sequence $\{f_1(q_{21}) = x_{11}(\text{say}), f_2(q_{22}) = x_{12}, ..., f_n(q_{2n}) = x_{1n}\} \in X_1$ and $\{g_1(q_{21}) = y_{11}(\text{say}), (g_2(q_{22}) = y_{12}, ..., g_n(q_{2n}) = y_{1n}\} \in Y_1$ such that $\forall i = 1, ..., n, \mu_1(q_{1i-1},\ x_{1i}, q_{1i}, y_{1i}) > 0$. Now consider the sequence $\{q_2 = q_{20}, q_{21}, ..., q_{2n} = p_2\}$ and the sequence $\{x_{21}, x_{22}, ..., x_{2n}\} \in X_2$ and $\{y_{21}, y_{22}, ..., y_{2n}\} \in Y_2$ such that $\forall i = 1, ..., n, \mu_2(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) > 0$.

The converse of the above theorem is true when individual fmms are strongly connected.

Theorem 3.13 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a strongly connected fmms, $i = 1, 2$. Then $M_1 \odot M_2$ is connected,where $\odot$ is restricted direct product, cascade product and wreath product.

Remark 3.4 Let $M_i = (Q_i, X_i, Y_i, \mu_i)$ be a fmms. Then

(1) by Theorem (2.13),(3.8) to (3.13) fmm $M_1 \odot M_2$ is strongly connected if and only if M_1 and M_2 are strongly connected , where $\odot$ is cartesian product, full direct product restricted direct product, cascade product and wreath product.

(2) by Theorem (2.4) the topology induced by $M_1 \odot M_2$ is discrete if and only if the topologies induced by M_1 and M_2 are discrete.

4. CONCLUSION

This paper is the study of fuzzy Mealy machines via topology, τ, defined by the successor function on their set of states. For this purpose, various kinds of fuzzy Mealy machines such as cyclic, retrievable, strongly connected and connected are introduced. These kinds of fuzzy Mealy machines are discussed with the help of this topology. The main findings of this paper are:
(i) The fuzzy Mealy machine M is strongly connected if and only if τ is the discrete topology on state set of M.
(ii)The fuzzy Mealy machine M is connected if and only if it has no separated submachine.
(iii) The cartesian product (full direct product) of two fuzzy Mealy machines is cyclic (respectively retrievable, union of strongly connected submachines and connected) if and only if they are individually cyclic (respectively retrievable, union of strongly connected submachines and connected).
(iv) If the restricted direct (cascade, wreath) product two fuzzy Mealy machines is cyclic (respectively retrievable, union of strongly connected submachines, connected) then they are individually cyclic (respectively retrievable, union of strongly connected submachines, connected). The converse of this result is true if both the fuzzy Mealy machines are strongly connected.
This paper will definitely leads to a study of decomposition as well as minimization of fuzzy Mealy machines from different angle, which may be the direction of further research. Moreover, one can introduce and study category of fuzzy Mealy machines. Also, various concepts from category theory as well as from fuzzy Mealy machines can be verified using the topology introduced in this paper.
This paper is concluded with the following open problem for further study. What kind of topologies are generated by the products of fuzzy Mealy machines? In our opinion these topologies will be very much different from so far known topologies. However, one can guess the nature of these topologies with the help of the theorems proved in this paper (see remark 3.4(2) and definition 2.8).

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