Dynamics of Antifractals in Noor Orbit

Ashish
Department of Mathematics
M. D. University, Rohtak
124001, INDIA

Mamta Rani
Department of Computer Science
Central University of Rajasthan
305801, INDIA

Renu Chugh
Department of Mathematics
M. D. University, Rohtak
124001, INDIA

ABSTRACT

Interesting antifractals are involved in the dynamics of antipolynomials $z \to \bar{z}^m + c$, for $m \geq 2$. The purpose of this paper is to visualize antifractals in Noor orbit and study the pattern among them.

Keywords: Antipolynomial, antifractal, tricorn, multicorn, antiJulia set, four-step feedback process, Noor orbit.

1. INTRODUCTION

The dynamics of antiholomorphic complex polynomials $z \to \bar{z}^m + c$, for $m \geq 2$, leads to interesting tricorns and multicorns antifractals with respect to one-step feedback process [4], two step-feedback process [9, 10] and three-step feedback process [2]. Tricorn prints are being used for commercial purpose, e.g. tricorn mugs and tricorn T shirts [13].

The polynomials $z \to \bar{z}^m + c$, for $m \geq 2$, have been studied mathematically using one-step feedback process. In 1989, Crowe et. al. [3] considered it as an formal analogy with Mandelbrot sets and named it as Mandelbar set and also brought their bifurcation features along arcs rather than at points. Multicorns have been found in a real slice of the cubic connectedness locus (cf. [6]). Winter showed that the boundary of the tricorn contains arc [12]. The symmetries of tricorn and multicorns have been analyzed by Lau and Schleicher [5], and Nakane and Schleicher [6] presented their various properties along with beautiful figures and quoted that multicorns are the generalized tricorns or the tricorns of higher order.

The purpose of this paper is to visualize tricorns and multicorns using four-step feedback process via Noor orbit and analyze them.

2. PRELIMINARIES

Definition 1 (Multicorn). The multicorns A_c for the quadratic function $A_c(z) = \bar{z}^m + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$A_c = \{c \in C : A_c^n(0) \text{ do not tend to} \to \infty\},$$

where C is a complex space, A_c^n is the n^{th} iterate of the function $A_c(z)$. An equivalent formulation is that the connectedness of loci for higher degree antiholomorphic polynomials $A_c(z) = \bar{z}^m + c$ are called multicorns [4].

Notice that at $m = 2$, multicorns reduce to tricorn. Moreover, the tricorns naturally lives in the real slice $d = \bar{c}$ in the two-dimensional parameter space of maps $z \to (z^2 + d)^2 + c$. They have $(m+1)$-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns [6].

Definition 2 (Julia Sets). The filled in Julia set of the function Q is defined as

$$K(Q) = \{z \in C : Q^k(z) \text{ does not tend to } \infty\},$$

where C is a complex space, $Q^k(z)$ is k'th iterate of function Q and $K(Q)$ denotes the filled in Julia set. The Julia set of the function Q is defined to be the boundary of $K(Q)$ i.e. $J(Q) = \partial K(Q)$, where $J(Q)$ denotes the Julia set. The set of points in $Q(z)$whose orbits are bounded under the Picard orbit is called the Julia set [8, p. 225].

Now, we give definition of Noor orbit, which will be used in the paper to implement four-step feedback process in the dynamics of $z \to \bar{z}^m + c$.

Definition 3 (Noor Orbit). Let us consider a sequence $\{x_n\}$ of iterates for initial point $x_0 \in X$ such that,

$$\{x_{n+1} : x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n;$$
$$y_n = (1-\beta_n)x_n + \beta_n T z_n;$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n;\ n = 0, 1 \ldots\},$$

where α_n, β_n, $\gamma_n \in [0, 1]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences convergent away from 0. The above sequence of iterates is called as Noor orbit, denoted by *NO*, which is a function of five tuples $(T, x_0, \alpha_n, \beta_n, \gamma_n)$ [7].

Notice that at $\gamma_n = 0$, *NO* reduces to Ishikawa orbit [2]; at $\beta_n = \gamma_n = 0$, *NO* reduces to Mann orbit (cf. [9, 10]); and at $\beta_n = \gamma_n = 0$ and $\alpha_n = 1$, it behaves as Picard orbit. In our further sections, we have chosen $\alpha_n = \alpha$, $\beta_n = \beta$ and $\gamma_n = \gamma$ to make the analysis simpler.

To visualize antifractals in *NO* for $z \to \bar{z}^m + c$, we shall require escape criterion with respect to *NO*. Escape criterion for $z \to \bar{z}^m + c$ in *NO* is $\max\{|c|, (2/\alpha)^{m-1}, (2/\beta)^{m-1}, (2/\gamma)^{m-1}\}$ [1].

3. MULTICORN IN *NO*

In this section, we programmed the polynomial $z \to \bar{z}^m + c$ in the software Mathematica 8.0 and tricorns and multicorns were generated in *NO* (see Figs. 1-12). Following are the observations made from generated multicorns:

- The number of branches in the tricorns and multicorns is $m+1$, where m is the power of $\bar{z}$. Also, few branches have m subranches (see Figs. 2-6).
- Multicorns exhibit $(m+1)$-fold rotational symmetries.

- For an m, there exist many multicorns.
- Higher degree multicorns become circular saw (Figs. 10-12). Rani [9, 10] had also given the similar conclusion while generating multicorns using two-step feedback process. The name circular saw was, first, given by Rani and Kumar to Mandelbrot sets [11].

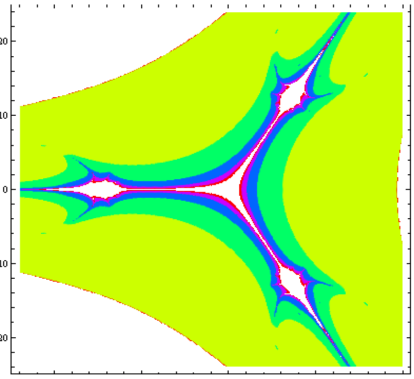

Fig. 1: Tricorn for m = 2, α = 0.01, β = γ = 0.1

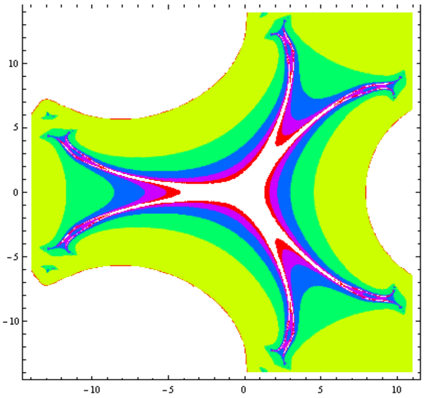

Fig. 2: Tricorn for m = 2, α = 0.1, β = 0.01, γ = 0.1

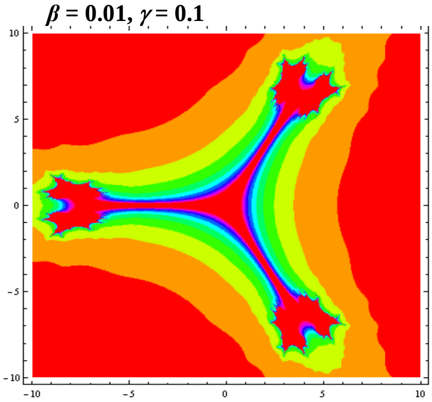

Fig. 3: Tricorn for m = 2, α = 0.1, β = 0.1, γ = 0.01

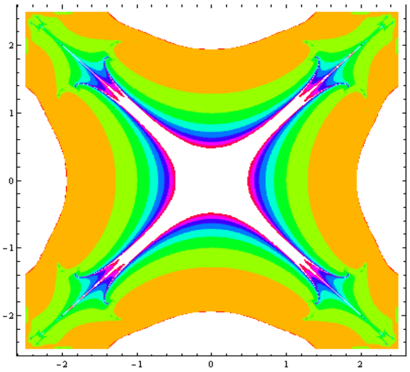

Fig. 4: Multicorn for m = 3, α = 0.1, β = γ = 0.1

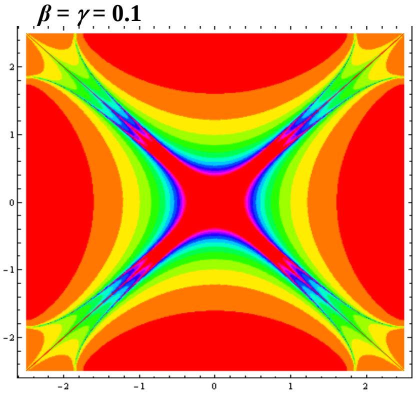

Fig. 5: Multicorn for m = 3, α = 0.05, β = 0.05, γ = 0.1

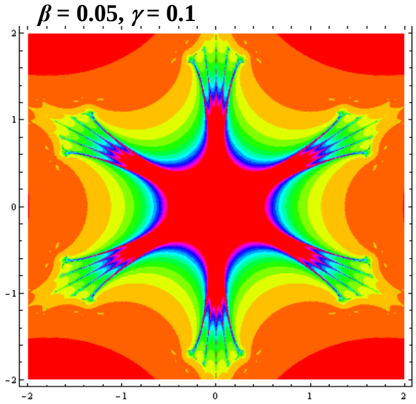

Fig. 6: Multicorn for m = 5, α = 0.01, β = 0.01, γ = 0.1

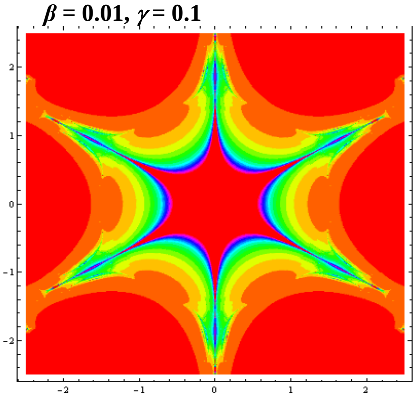

Fig.7: Multicorn for m = 5, α=0.01, β=0.1, γ=0.03

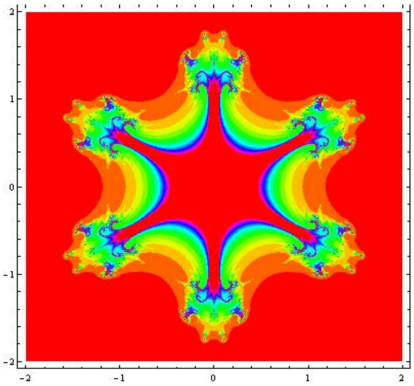

Fig. 8: Multicorn for m = 5, α=0.2, β=0.09, γ=0.1

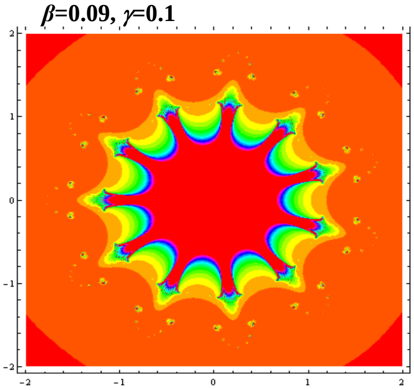

Fig. 9: Multicorn for m = 10, α = 0.01, β = 0.01, γ = 0.1

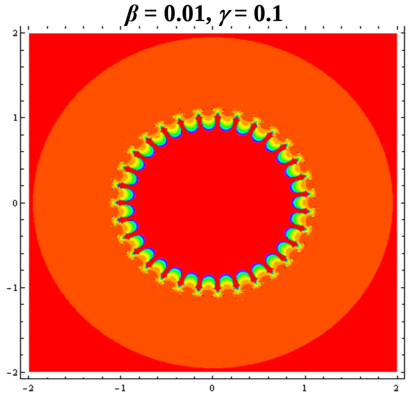

Fig. 10: Circular saw multicorn for m = 30, $\alpha = \beta$ = 0.01, γ = 0.1

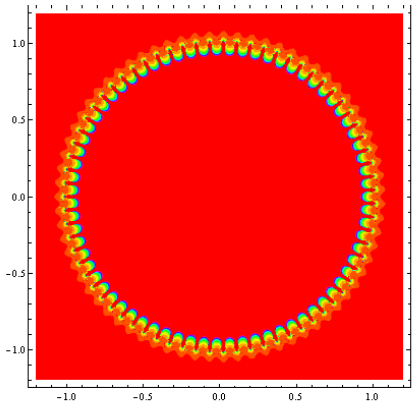

Fig. 11: Circular saw multicorn for m = 70, $\alpha = \beta$ = 0.01, γ = 0.1

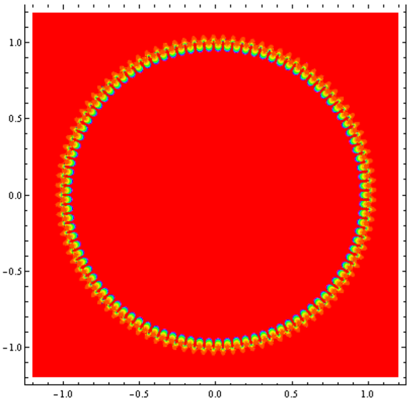

Fig. 12: Circular saw multicorn for m = 100, $\alpha = \beta$ = 0.01, γ = 0.1

4. ANTIJULIA SETS IN *NO*

Anti Julia sets have been generated for $z \longrightarrow \overline{z}^m + c$ in *NO* (see Figs. 13-21). Figs. 13-15 show that at m = 2, the anti Julia sets take the shape of tricorns. Further, it has been observed that the higher degree anti Julia sets become circular saw (Figs. 20 and 21).

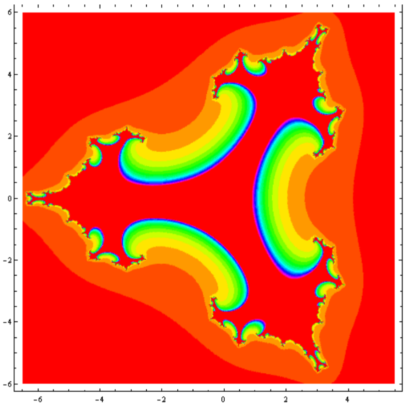

Fig. 13: AntiJulia set for m = 2, α = 0.4, $\beta = \gamma$ = 0.1, c = 0.2+0.2I

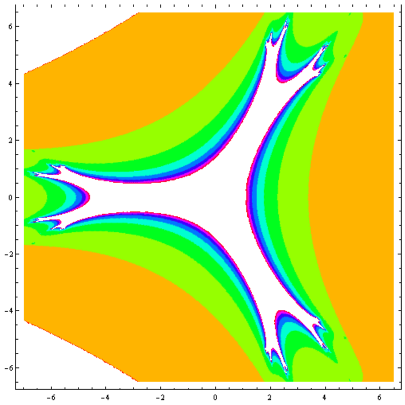

Fig. 14: AntiJulia set for m = 2, $\alpha = \beta$ = 0.05, γ = 0.3, c = 0.05+0.05I

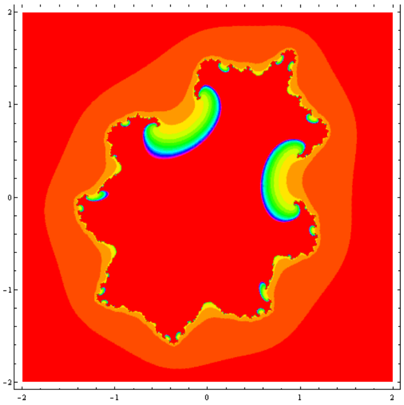

Fig. 15: AntiJulia set for m = 2, α = β = 0.9, γ = 0.2, c = 0.31+0 .55I

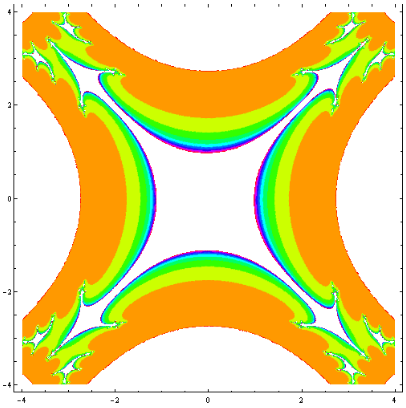

Fig. 16: AntiJulia set for m = 3, β = α = 0.05, γ = 0.1, c = 0.2+0.2I

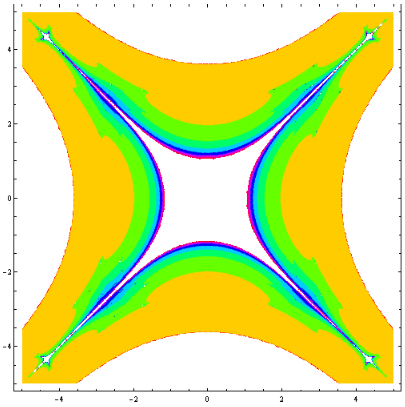

Fig. 17: AntiJulia set for m = 3, α = 0.015, β=γ=0.1, c=0.2+0.2I

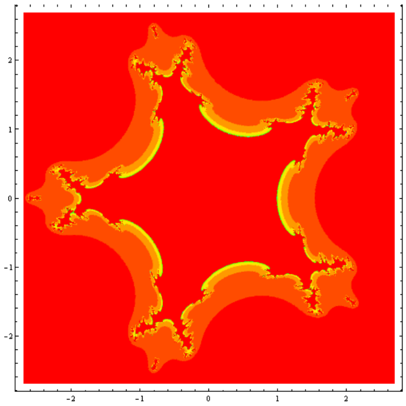

Fig. 18: AntiJulia set for m = 4, α = β = 0.1, γ = 0.3, c = 0.05+0.05I

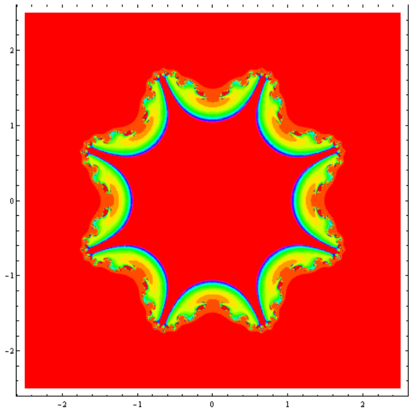

Fig. 19: AntiJulia set for m =7, α = 0.05, β = γ = 0.01, c = 0.1+1I

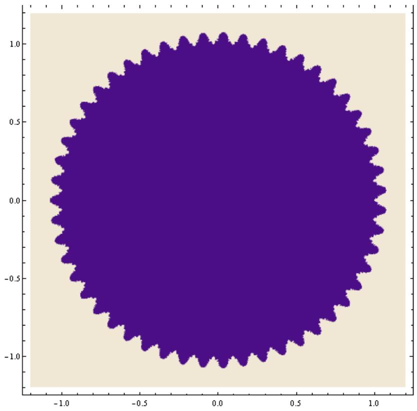

Fig. 20: Circular saw AntiJulia set fo m = 50, α = β = 0.05, γ = 0.3, c = 0.05+0.05I

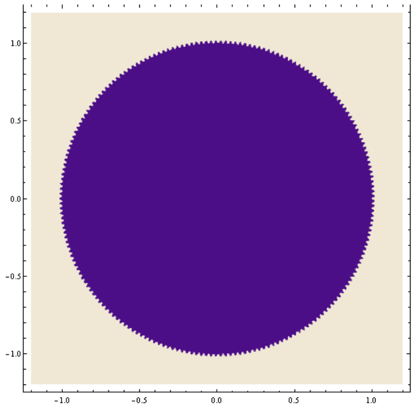

Fig. 21: Circular saw AntiJulia for $m = 200$, $\alpha = \beta = .05$, $\gamma = .3$, $c = .05 + .05I$

5. CONCLUSION

In the dynamics of antipolynomials $z \longrightarrow \bar{z}^m + c$, where $m \geq 2$, there exist many multicorns for the same value of m in Noor orbit. AntiJulia sets have also been generated in Noor orbit. Further, it was found that for higher degrees of the polynomial, all the antifractals become circular saw.

6. ACKNOWLEDGMENTS

This Research is supported by the University Grant Commission of India (Grant No. 39-29/2010(SR))

7. REFERENCES

[1] Ashish, M. Rani, and R. Chugh, New Julia sets and Mandelbrot sets in Noor orbit, (preprint), 2012.

[2] Y. S. Chauhan, R. Rana, and A. Negi, New tricorn and multicorn of Ishikawa iterates, Int. J. Comput. Appl., (7)(13)(2010), 25-33.

[3] W. D. Crowe, R. Hasson, P. J. Rippon, and P. E. D. Strain-Clark, On the structure of the Mandelbar set, Nonlinearity, (2)(4)(1989), 541-553.

[4] R. L. Devaney, A first course in chaotic dynamical systems: theory and experiment, Addison-Wesley, New York, 1992.

[5] E. Lau, and D. Schleicher, Symmetries of fractals revisited. Math. Intelligencer (18)(1)(1996), 45-51.

[6] S. Nakane, and D. Schleicher, On multicorns and unicorns: I. Antiholomorphic dynamics hyperbolic components and real cubic polynomials, Int. J. Bifur. Chaos Appl. Sci. Engr., (13)(10)(2003), 2825-2844.

[7] M. A Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., (251)(2000), 217-229.

[8] H. O. Peitgen, H. Jurgens, and D. Saupe, Chaos and Fractals, Springer-Verlag, New York, 1994.

[9] M. Rani, Superior antifractals, *in*: IEEE Proc. ICCAE 2010, vol. 1, 798-802.

[10] M. Rani, Superior tricorns and multicorns, *in:* Proc. 9th WSEAS Int. Conf. on Appl. Comp. Engg. (ACE'10), 2010, 58-61.

[11] M. Rani, and M. Kumar, Circular saw Mandelbrot sets, *in:* Proc. 14th WSEAS Int. Conf. on Appl. Math.(Math'09), 2009, 131-136.

[12] R. Winters, Bifurcations in families of antiholomorphic and biquadratic maps, Ph.D Thesis, Boston Univ., London, 1990.

[13] Zazzle, http://www.zazzle.com/tricorn+gifts