Edge Dominating Functions of Quadratic Residue Cayley Graphs

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ABSTRACT

The concept of edge domination is introduced by Mitchell and Hedetniemi [6]. Further results on edge domination are given in Arumugam and Velammal [2]. Functional generalization for vertex subsets has been studied extensively in literature [4, 5]. Cockayne and Mynhardt [3] have introduced that edge subsets may also be embedded into sets of functions and an analogous concept of convexity could also be developed. In this paper we obtain results on minimal edge dominating functions of $G(Z_p, Q)$ and the convexity of these functions are discussed. The theory of Edge Dominating Functions in quadratic residue Cayley graphs helps in finding optimal global and local alignments for the smooth conduction of a work and improves the ability of a task or a job in connected systems such as transportation process, communication tools, networks etc.

Keywords

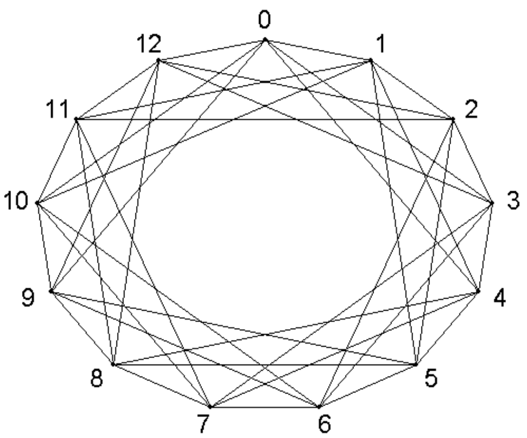
Edge Dominating Functions – Minimal Edge Dominating Functions – Convexity.

1. INTRODUCTION

The quadratic residue Cayley graph $G(Z_p, Q)$, that is, the Cayley graph associated with the set of quadratic residues modulo an odd prime p, is defined as follows.

Let p be an odd prime, S, the set of quadratic residues modulo p and let $S^* = \{ s, p - s$ / s ⍉ S, s ◷ p}. The **quadratic residue Cayley graph** $G(Z_p, Q)$ is defined as the graph whose vertex set is $Z_p = \{0, 1, 2, 3, \ldots\ldots, (p-1)\}$ and the edge set $E = \{ (x, y) / x - y$ or $y - x$ is in $S^*\}$.

Example: The Graph $G(Z_p, Q)$ for p = 13 is given below.



We now consider the definitions of edge dominating function, minimal edge dominating function.

Edge Dominating Set: Let G(V, E) be a graph. A subset F of E is called an edge dominating set (EDS) if each edge in E – F is adjacent to atleast one edge in F.

Minimal Edge Dominating Set: Let G(V, E) be a graph. An EDS, F is called a minimal edge dominating set (MEDS) if no proper subset of F is an EDS of G.

Edge Dominating Function: Let G(V, E) be a graph. A function $f : E \to [0,1]$ is called an edge dominating function (EDF) if $\sum_{e' \in N[e]} f(e') \geq 1$ for all $e \in E(G)$, where N[e] is the closed neighbourhood of the edge e.

Minimal Edge Dominating Function: An EDF f is called a minimal edge dominating function (MEDF), if for all functions $g : E \to [0,1]$ with $g < f$, g is not an edge dominating function.

Let G(V, E) be a graph and A, B $\subseteq$ E. We say that **A dominates B** if every edge in B – A is adjacent to an edge in A and we write $A \to B$.

Let f be any EDF of G. The **boundary set** B'_f and the **positive set** P'_f of f are defined by

$$B'_f = \left\{ e \in E : \sum_{e' \in N[e]} f(e') = 1 \right\}$$

and $$P'_f = \{ e \in E : f(e) > 0 \}.$$

We need the following result referred from Arumugam and Sitara Jerry [1].

Theorem 1.1: An EDF f is a MEDF of G if and only if $B'_f \to P'_f$.

2. MAIN RESULTS

Theorem 2.1: Let F be a MEDS of $G(Z_p, Q)$. Then a function $f : E \to [0,1]$ defined by

$$f(e) = \begin{cases} 1, & if\ e \in F, \\ 0, & otherwise. \end{cases}$$

becomes a MEDF of $G(Z_p, Q)$.

Proof: Consider $G(Z_p, Q)$. Let F be a MEDS of $G(Z_p, Q)$.

Let $|F| = n > 1$. Let f be the function defined as in the hypothesis. The summation value taken over the neighbourhood N[e] of e ⍉ E is

$$\sum_{e' \in N[e]} f(e') = \begin{cases} 1, & if\ any\, one\, edge\, of\ F\ is\ in\ N[e] \\ s \times 1, & if\ s\ edges\ of\ F\ are\, in\, N[e] \end{cases}$$

where s < n and s > 1.

Thus $\sum_{e' \in N[e]} f(e') \geq 1$, ✂ e ⍉ E.

Therefore f is an EDF.

We now check for the minimality of f.

Define $g : E \to [0,1]$ by

$$g(e) = \begin{cases} r, & if\ e = e_i \in F, \\ 1, & if\ e \in F - \{e_i\}, \\ 0, & otherwise. \end{cases}$$

where 0 < r < 1.

Since strict inequality holds at the edge e ⍉ F of E, it follows that g < f.

Now

$$\sum_{e' \in N[e]} g(e') = \begin{cases} r, & if\ e_i \in F\ is\ in\ N[e], \\ 1, & if\ any\ one\ edge\ e\ of\ F - \{e_i\}\ is\ in\ N[e], \\ s \times 1, & if\ s\ edges\ of\ F - \{e_i\}\ are\ in\ N[e]. \end{cases}$$

So $\sum_{e' \in N[e]} g(e') \ngeq 1$,✂ e ⍉ E.

This implies that g is not an EDF. Since r < 1 is arbitrary it follows that there exists no g < f such that g is an EDF.

Thus f is a MEDF of G(Z_p, Q). ■

Theorem 2.2: Let f_1 and f_2 be two distinct MEDFs of G(Z_p , Q) defined from E to [0, 1]. Let h be the convex combination of f_1 and f_2. Then $B'_h = B'_{f_1} \cap B'_{f_2}$ and $P'_h = P'_{f_1} \cup P'_{f_2}$.

Proof: Consider G(Z_p, Q). Let f_1 and f_2 be two distinct MEDFs. Let $h = \lambda_1 f_1 + \lambda_2 f_2$, where $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, and $\lambda_1 + \lambda_2 = 1$.

We now claim that $B'_h = B'_{f_1} \cap B'_{f_2}$ and $P'_h = P'_{f_1} \cup P'_{f_2}$.

Let $e \in B'_{f_1} \cap B'_{f_2}$.

Then $\sum_{e' \in N[e]} f_i(e') = 1, \quad for\ \ i = 1, 2.$

Now

$$\begin{aligned} \sum_{e' \in N[e]} h(e') &= \sum_{e' \in N[e]} \left[\sum_{i=1}^{2} \lambda_i f_i \right](e') \\ &= \sum_{e' \in N[e]} [\lambda_1 f_1(e') + \lambda_2 f_2(e')] \\ &= \lambda_1 \sum_{e' \in N[e]} f_1(e') + \lambda_2 \sum_{e' \in N[e]} f_2(e') \\ &= \lambda_1 .1 + \lambda_2 .1 = \lambda_1 + \lambda_2 = 1. \end{aligned}$$

Therefore $\sum_{e' \in N[e]} h(e') = 1$. This implies that $e \in B'_h$.

Hence it follows that $B'_{f_1} \cap B'_{f_2} \subseteq B'_h$. ------------ (1)

Suppose $e \notin B'_{f_1}$ or $e \notin B'_{f_2}$.

Then $\sum_{e' \in N[e]} f_1(e') > 1$ or $\sum_{e' \in N[e]} f_2(e') > 1$.

Now

$$\begin{aligned} \sum_{e' \in N[e]} h(e') &= \sum_{e' \in N[e]} \left[\sum_{i=1}^{2} \lambda_i f_i \right](e') \\ &= \lambda_1 \sum_{e' \in N[e]} f_1(e') + \lambda_2 \sum_{e' \in N[e]} f_2(e') \\ &> \lambda_1 .1 + \lambda_2 .1 = \lambda_1 + \lambda_2 = 1. \end{aligned}$$

Therefore $\sum_{e' \in N[e]} h(e') > 1$.

This implies that $e \notin B'_h$.

Thus it follows that $B'_h \subseteq B'_{f_1} \cap B'_{f_2}$. ------------- (2)

Hence from (1) and (2), we get $B'_h = B'_{f_1} \cap B'_{f_2}$.

Now let $e \in P'_{f_1} \cup P'_{f_2}$.

Then either $f_1(e) > 0$ or $f_2(e) > 0$.

So $\lambda_1 f_1(e) + \lambda_2 f_2(e) > 0$ and hence $h(e) > 0$.

This implies that $e \in P'_h$.

Thus $P'_{f_1} \cup P'_{f_2} \subseteq P'_h$. ------------- (3)

Suppose $e \notin P'_{f_1} \cup P'_{f_2}$. Then $e \notin P'_{f_1}$ and $e \notin P'_{f_2}$.

This implies $f_1(e) = 0$ and $f_2(e) = 0$.

Hence it follows that $h(e) = 0$.

This implies that $e \notin P'_h$

Thus $P'_h \subseteq P'_{f_1} \cup P'_{f_2}$. ------------- (4)

Hence from (3) and (4), it follows that $P'_{f_1} \cup P'_{f_2} = P'_h$.

Therefore the theorem follows. ■

Theorem 2.3: Let $f_1 : E \to [0,1]$ and $f_2 : E \to [0,1]$ be two distinct MEDFs of G(Z_p, Q). The convex combination of f_1 and f_2 is a MEDF if and only if $B'_{f_1} \cap B'_{f_2} \to P'_{f_1} \cup P'_{f_2}$ in G(Z_p, Q).

Proof: Consider G(Z_p, Q). Let f_1 and f_2 be two distinct MEDFs.

Let $h = \lambda_1 f_1 + \lambda_2 f_2$, where $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, and $\lambda_1 + \lambda_2 = 1$.

By Theorem 2.2 we have $B'_h = B'_{f_1} \cap B'_{f_2}$ and $P'_h = P'_{f_1} \cup P'_{f_2}$. Now by Theorem 1.1, h is a MEDF if and only if $B'_h \to P'_h$. Hence it follows that h is a MEDF if and only if $B'_{f_1} \cap B'_{f_2} \to P'_{f_1} \cup P'_{f_2}$ in G(Z_p, Q). ■

Theorem 2.4: Let $f_1, f_2, \ldots, f_n$ be n distinct MEDFs of G(Z_p, Q). The convex combination of $f_1, f_2, \ldots, f_n$ is a

MEDF if and only if
$B'_{f_1} \cap B'_{f_2} \cap \ldots \cap B'_{f_n} \to P'_{f_1} \cup P'_{f_2} \cup \ldots \cup P'_{f_n}$.

Proof: Let $f_1, f_2, \ldots, f_n$ be n distinct MEDFs of G(Z_p, Q).

Let h be a convex combination of $f_1, f_2, \ldots, f_n$,

i.e., $h = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$,

where $0 < \lambda_i < 1,\ (i = 1, 2, 3, \ldots, n)$ and $\sum_{i=1}^{n} \lambda_i = 1$.

We first claim that $B'_h = B'_{f_1} \cap B'_{f_2} \cap \ldots \cap B'_{f_n}$ and $P'_h = P'_{f_1} \cup P'_{f_2} \cup \ldots \cup P'_{f_n}$.

Let $e \in B'_{f_1} \cap B'_{f_2} \cap \ldots \cap B'_{f_n}$.

Then $\sum_{e' \in N[e]} f_i(e') = 1, \quad for \ \ i = 1, 2, \ldots, n.$

Now

$$\sum_{e' \in N[e]} h(e') = \sum_{e' \in N[e]} \left[\sum_{i=1}^{n} \lambda_i f_i \right] (e')$$
$$= \sum_{e' \in N[e]} [\lambda_1 f_1(e') + \lambda_2 f_2(e') + \ldots\ldots + \lambda_n f_n(e')]$$
$$= \lambda_1 \sum_{e' \in N[e]} f_1(e') + \lambda_2 \sum_{e' \in N[e]} f_2(e') + \ldots + \lambda_n \sum_{e' \in N[e]} f_n(e')$$
$$= \lambda_1.1 + \lambda_2.1 + \ldots + \lambda_n.1$$
$$= \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1.$$

This implies that $e \in B'_h$.

Hence $\bigcap_{i=1}^{n} B'_{f_i} \subseteq B'_h$. -------------- (1)

Suppose $e \notin B'_{f_i}$ for some i. Then $\sum_{e' \in N[e]} f_i(e') > 1$.

So $\sum_{e' \in N[e]} h(e') > 1$. This implies that $e \notin B'_h$.

Thus it follows that $B'_h \subseteq \bigcap_{i=1}^{n} B'_{f_i}$. -------------- (2)

Hence from (1) and (2) we get $B'_h = \bigcap_{i=1}^{n} B'_{f_i}$.

Now let $e \in \bigcup_{i=1}^{n} P'_{f_i}$.

This implies that $e \in P'_{f_i}$ for some i, so that $f_i(e) > 0$.

Hence it follows that h(e) > 0,

since $h(e) = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n > 0.$

This implies that $e \in P'_h$.

Therefore we have $\bigcup_{i=1}^{n} P'_{f_i} \subseteq P'_h$ -------------- (3)

Suppose if $e \notin \bigcup_{i=1}^{n} P'_{f_i}$, then $e \notin P'_{f_i}$ for each i.

Hence it follows that $f_i(e) = 0$, for each i.

This implies that h(e) = 0.

Thus we have $P'_h \subseteq \bigcup_{i=1}^{n} P'_{f_i}$ --------------- (4)

From (3) and (4), it follows that $P'_h = \bigcup_{i=1}^{n} P'_{f_i}$.

Now by Theorem 1.1, it follows that h is a MEDF if and only if $B'_h \to P'_h$.

Hence h is a MEDF if and only if $\bigcap_{i=1}^{n} B'_{f_i} \to \bigcup_{i=1}^{n} P'_{f_i}$. ■

Lemma 2.5: If the convex combination of a collection of MEDFs $f_1, f_2, \ldots, f_n$ is minimal in G(Z_p, Q), then the convex combination of any proper sub-collection of $f_1, f_2, \ldots, f_n$ is also minimal in G(Z_p, Q).

Proof: Let $f_1, f_2, \ldots, f_n$ be the collection of MEDFs of G(Z_p, Q). Suppose the convex combination of $f_1, f_2, \ldots, f_n$ is minimal. Then by Theorem 2.4, it follows that $B'_{f_1} \cap B'_{f_2} \cap \ldots \cap B'_{f_n} \to P'_{f_1} \cup P'_{f_2} \cup \ldots \cup P'_{f_n}$.

Let $f'_1, f'_2, \ldots, f'_m$, where m < n, be a sub-collection of $f_1, f_2, \ldots, f_n$. Then $\bigcap_{j=1}^{m} B'_{f'_j} \supseteq \bigcap_{i=1}^{n} B'_{f_i}$ and $\bigcup_{j=1}^{m} P'_{f'_j} \subseteq \bigcup_{i=1}^{n} P'_{f_i}$.

Since $\bigcap_{i=1}^{n} B'_{f_i} \to \bigcup_{i=1}^{n} P'_{f_i}$, it follows that $\bigcap_{j=1}^{m} B'_{f'_j} \to \bigcup_{j=1}^{m} P'_{f_j}$.

Thus the convex combination of $f'_1, f'_2, \ldots, f'_m$ is a MEDF of G(Z_p, Q). ■

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