A New Class αcg-set Weaker Form of Closed Sets in Topological Spaces

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ABSTRACT
In this paper, a new class αcg-closed sets, as weaker forms of closed sets in topological spaces are introduced . Some of its properties are studied. Also we have provided continuity ,closed map and open map are also introduced .

Keywords
αcg-closed, αcg- continuous functions, αcg closed map, αcg open map.

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1. INTRODUCTION

In 1970, Levine [1] first considered the concept of generalized closed (briefly,g-closed) sets were defined and investigated. Tong [8] and Hatir E [4] introduced B-sets and t-sets and α*-sets respectively. t-sets and α*-sets are weak forms of open sets. In this paper, we have introduced a new class of sets called αcg-closed sets and study some of their properties. Noiri,[21] introduced α-continuous functions. Sundaram [6] introduced the concept of generalized continuous function includes the class of continuous functions and studies several properties related to it. In this section, we introduce the concepts of αcg-continuous function in topological spaces and study some of their properties. It is an extension study of [20]for continuous functions, closed and open maps.

2. PRELIMINARIES

Definition: 2.1 A subset A of a topological space(X, τ) is called

(i)Generalized closed (g-closed)[1] if cl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is open in X.

(ii)Semi-generalized closed(sg-closed)[9] if scl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is semi open in X.

(iii)Generalized semi preclosed [12] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is open in X.

(iv)Weakly generalized closed[10] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U, and U is open in X.

(v)α-generalized closed αg-closed[15] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is open in X.

Definition: 2.2 A subset S of X is called

(i) semiopen [1] if there exists an open set G such that G $\subseteq$ S $\subseteq$ cl(G) and semi closed[13] if there exists a closed set F such that int(F) $\subseteq$ S $\subseteq$ F.

(ii) α-open[16] if S $\subseteq$ int(cl(int(S))) and α-closed if S $\supseteq$ cl(int(cl(S))).

Result: 2.3 (i) αcl(S)=S $\cup$ cl(cl(int(cl(S)))[14].
(ii) α int(S)= S $\cap$ int(cl(int(S)))[11].
(iii) In this work, we use the notations C(S) sets for C-sets (Due to Sundaram)[7].

Definition: 2.4 For a subset A of X is called
(a) a t-set in X[8] if int(A) = int(cl(A)).

(b) a B-set in X[8] if A = G $\cap$ F where G is open and F is a t-set in X.

(c) an α*-set in X[4] if int(A) = int(cl(int(A))).

(d) a C-set(Due to Sundaram)[7] if A= G $\cap$ F where G is g-open and F is a t-set in X.

(e) a C-set (Due to Hatir, Noiri and Yuksel)[4] if A = G $\cap$ F where G is open and F is an α*-set in X.

(f) a C*set[5] if A= G $\cap$ F where G is g-open and F is an α*-set in X.

Definition: 2.5 A map f: X →Y is called
(a) semicontinuous[1] if f^{-1}(F) is semiclosed in X for each closed set F in Y,

(b) generalized continuous (g-continuous)[17] if f^{-1}(F) is

(c) g-closed in X for each closed set F in Y,

(d) α-generalized continuous (αg-continuous)[18] if f^{-1}(F) is αg-closed in X for each closed set F in Y,

(e) generalized semicontinuous (gs-continuous)[6] if f^{-1}(F) is gs-closed in X for each closed set F in Y,

(f) α-continuous [19] if f^{-1}(F) is α-open in X for each open set F in Y.

(g) closed map if for each closed set F in X, f(F) is closed in Y.

(h) open map if for each open set F in X, f(F) is open in Y.

3. αcg-SET IN TOPOLOGICAL SPACES

Definition:3.1 A subset A of X is called αcg- set if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is C-set.

Remark:3.2 i) The complement of αcg- set is called αcg-open set.

Theorem:3.3 Every closed set in X is αcg- set in X but not conversely.

Proof: Let A be a closed set in X. Let U be a C-set such that A $\subseteq$ U. Since A is closed, cl(A)=A, therefore cl(A) $\subseteq$ U. But αcl(A) $\subseteq$ cl(A) $\subseteq$ A $\subseteq$ U. Therefore αcl(A) $\subseteq$ U. Hence A is αcg set in X.

The converse of the theorem 3.3 need not be true as seen from the following example.

Example:3.4 Consider the topological space X={a,b,c} with the topology $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. The sets {b,c} is αcg set in X, but not closed in X.

Theorem:3.5 The union of two αcg- sets is αcg-set in X.

Proof: Let A and B are αcg- set in X. Let U be an αcg in X such that $A \cup B \subseteq U$ then $A \subseteq U$, $B \subseteq U$. SinceA and B are αcg-sets then αcl(A)$\subseteq$U, αcl(B)$\subseteq$U. Hence $\alpha\, cl(A \cup B) = \alpha\, cl(A) \cup \alpha\, cl(B) \subseteq U$. Therefore AUB is αcg- set in X.

Theorem:3.6 Every g-closed set in X is αcg- set in X but not conversely.

Proof: Let A be g-closed set in X. Let U be a C- set in X such that A $\subseteq$ U. To prove αcl(A) $\subseteq$ U. Since U is C-set, U=P∩Q where P is open, Q is α* set. Therefore U $\subseteq$ P, U $\subseteq$ Q. Therefore A $\subseteq$ U $\subseteq$ P implies A $\subseteq$ P , P is open since A is g-closed implies cl(A) $\subseteq$ P. Since $\alpha\, cl(A) \subseteq cl(A) \subseteq U \subseteq P$ .That is αcl(A) $\subseteq$ U.

The converse of the theorem 3.6 need not be true as seen from the following example

Example:3.7 Consider the topological space X={a,b,c} with the topology $\tau = \{\phi, X, \{a\}\}$. The set {a } is αcg-set in X, but not g-closed in X.

Theorem:3.8 Every αg-closed set in X is αcg- set in X but not conversely.

Proof: Let A be αg-closed. Let U be a C- set in X such that A $\subseteq$ U. To prove αcl(A) $\subseteq$ U. Since U is C-set, U=P∩Q where P is open, Q is α* set. Therefore U $\subseteq$ P, U $\subseteq$ Q.Since A $\subseteq$ U $\subseteq$ P implies A $\subseteq$ P , P is open since A is αg-closed implies cl(A) $\subseteq$ P. Since $\alpha\, cl(A) \subseteq cl(A) \subseteq U \subseteq P$ .That is αcl(A) $\subseteq$ U.

The converse of the theorem 3.8 need not be true as seen from the following example

Example:3.9 Consider the topological space X={a,b,c} with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. The sets {a,b} is αcg set in X, but not αg-closed in X.

Theorem:3.10 Every gs-closed set in X is αcg- set in X but not conversely.

Proof: Let A be gs-closed. Let U be a C- set in X such that A $\subseteq$ U. To prove αcl(A) $\subseteq$ U. Since U is C-set, U=P∩Q where P is open, Q is α* set. Therefore U $\subseteq$ P, U $\subseteq$ Q. Since A $\subseteq$ U $\subseteq$ P implies A $\subseteq$ P , P is open.Since A is gs-closed implies cl(A) $\subseteq$ P. Since $s\,cl(A) \subseteq cl(A) \subseteq U \subseteq P$, αcl(A) $\subseteq$ cl(A) $\subseteq$ U $\subseteq$ P. That is αcl(A) $\subseteq$ U.

The converse of the theorem 3.10 need not be true as seen from the following example

Example:3.11 Consider the topological space X={a,b,c} with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. The sets {a,b} is αcg set in X, but not gs-closed in X.

Theorem:3.12 Every α-closed set in X is αcg- set in X but not conversely.

Proof: Let A be α-closed. Let U be a C- set in X such that A $\subseteq$ U. To prove αcl(A) $\subseteq$ U. Since A is α closed set cl(int(cl(A))) $\subseteq$ U implies $A \cup cl(\text{int}(cl(A))) \subseteq U$. That is αcl(A) $\subseteq$ U.

The converse of the theorem 3.12 need not be true as seen from the following example

Example:3.13 Consider the topological space X={a,b,c} with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. The sets {a,b} is αcg set in X,but not α-closed in X.

We illustrate the relations between various sets in the following diagram Fig 3.1.

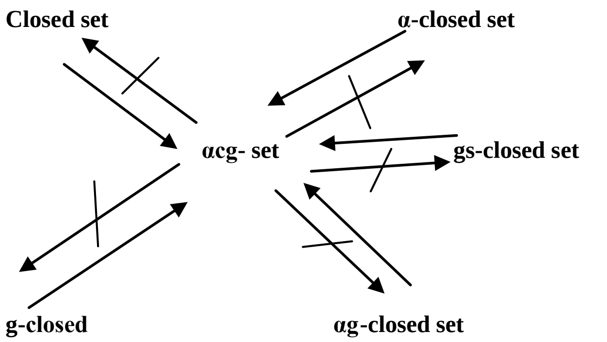

Fig:3.1

In the above diagram Fig 3.1 none of the implications can be reversed.

Remark:3.14 The concept of αcg- set is independent of the following classes of sets namely semiclosed sets, β-closed, weakly g-closed.

Example:3.15 Consider the topological space X={a,b,c} with topology $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. In this space, the sets {b},{c} are β-closed,weakly g-closed in X, but not αcg sets in X. The {b,c} is αcg set but not β-closed, weakly g-closed in X.

Example:3.16 Consider the topological space X={a,b,c} with topology $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. In this space, the sets {b},{c} are semiclosed but not αcg sets and {a} is αcg sets but not semiclosed set in X.

From above theorem and example . We get the following diagram Fig 3.2 with

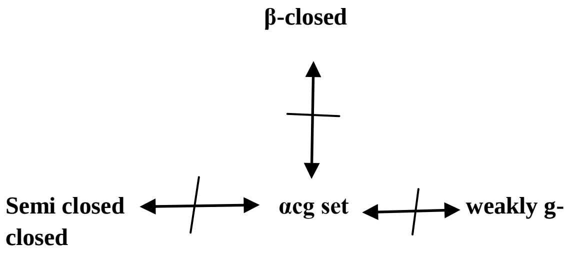

Fig 3.2

4. αcg CONTINUOUS MAPPINGS IN TOPOLOGICAL SPACES

In this section we introduce αcg continuous mappings.

Definition:4 A map f;X→Y from topological space X into a topological space Y is called αcg continuous if its inverse image of every closed in Y is αcg closed in X.

Theorem:4.0 If a map f:X→Y is continuous then it is αcg continuous but not conversely.

Proof: Let f:X→Y be continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}[F]$ is closed in Y. Since every closed set is αcg closed set, $f^{-1}[F]$ is αcg closed in X. Therefore f is αcg continuous.

The converse of the theorem 4.0 need not be true as seen from the following example

Example:4.1 Let X=Y={a,b,c} , τ={ϕ,X,{b},{c},{b,c}} , σ={ϕ,Y,{a}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is αcg continuous but not continuous, since for the closed set {b,c} in Y, f^{-1}({b,c})={b,c} is not closed in X, Therefore f is not continuous.

Theorem:4.2 If a map f:X→Y is g-continuous then it is αcg continuous but not conversely.

Proof: Let f:X→Y be g-continuous. Let F be any g-closed set in Y. Then the inverse image $f^{-1}[F]$ is g-closed in Y. Since every g-closed set is αcg closed set , $f^{-1}[F]$ is αcg closed set in X. Therefore f is αcg continuous.

The converse of the theorem 4.2 need not be true as seen from the following example

Example:4.3 Let X=Y={a,b,c} , τ={ϕ,X,{a}} , σ={ϕ,Y,{b,c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is αcg continuous but not g-continuous since for the closed set {a} in Y, f^{-1}({a})={a} is not g-closed in X.Therefore f is not
g-continuous.

Theorem:4.4 If a map f:X→Y is αg-continuous then it is αcg continuous but not conversely.

Proof: Let f:X→Y be αg-continuous. Let F be any αg-closed set in Y. Then the inverse image $f^{-1}[F]$ is αg-closed in Y. Since every αg-closed set is αcg closed set, $f^{-1}[F]$ is αcg closed set in X. Therefore f is αcg continuous.

The converse of the theorem 4.4 need not be true as seen from the following example.

Example:4.5 Let X=Y={a,b,c} , τ={ϕ,X,{a},{b},{a,b}} , σ={ϕ,Y,{c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is αcg continuous but not αg-continuous since for the closed set {a,b} in Y, f^{-1}({a,b})={a,b} is not αg-closed in X. Therefore f is not αg-continuous.

Theorem:4.6 If a map f:X→Y is α-continuous then it is αcg continuous but not conversely.

Proof: Let f:X→Y be α-continuous. Let F be any α-closed set in Y. Then the inverse image $f^{-1}[F]$ is α-closed in Y. Since every αclosed set is αcg closed set, $f^{-1}[F]$ is αcg closed set in X. Therefore f is αcg continuous.

The converse of the theorem 4.6 need not be true as seen from the following example.

Example:4.7 Let X=Y={a,b,c} , τ={ϕ,X,{a},{b},{a,b}} , σ={ϕ,Y,{c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is αcg continuous but not α-continuous since for the closed set {a,b} in Y, f^{-1}({a,b})={a,b} is not α-closed in X.Therefore f is not α-continuous

Theorem:4.8 If a map f:X→Y is gs-continuous then it is αcg continuous but not conversely.

Proof: Let f:X→Y be gs-continuous. Let F be any gs-closed set in Y. Then the inverse image $f^{-1}[F]$ is

gs-closed in Y. Since every gs-closed set is αcg closed set, $f^{-1}[F]$ is αcg closed set.. Therefore f is αcg continuous.

The converse of the theorem 4.8 need not be true as seen from the following example.

Example:4.9 Let X=Y={a,b,c} , τ={ϕ,X,{c}} , σ={ϕ,Y,{a},{b},{a,b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then f is αcg continuous but not gs-continuous since for the closed set {c} in Y, f^{-1}({c})={c} is not gs-closed in X.

We illustrate the relations between various generalizations of continuous functions in the following diagram Fig 4.1.

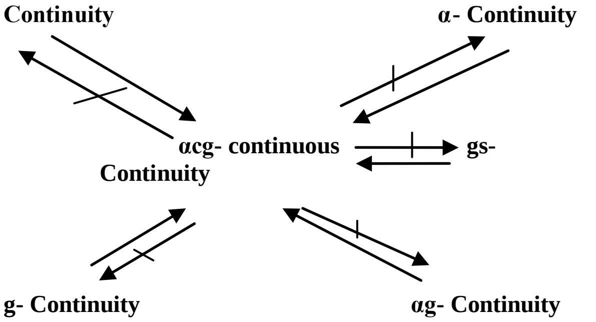

Fig 4.1

In the above diagram Fig 4.1 none of the implications can be reversed.

5. αcg-CLOSED MAPS AND αcg-OPEN MAPS IN TOPOLOGICAL SPACES

Malghan [22] introduced and investigated some properties of generalized closed maps in topological spaces. The concept of generalized open map was introduced by Sundaram [6]. Biswas [23]defined semi open mappings as a generalization of open mappings and studied several of its properties.

In this section, we introduced the concepts of αcg-closed maps and αcg -open maps in topological spaces.

Definition :5.1 A map f : X →Y is called αcg -closed map if for each closed set F in X, f(F) is a αcg -closed set in Y.

Theorem : 5.2 If f : X →Y is a closed map, then it is
αcg -closed but not conversely.

Proof : Since every closed set is αcg -closed the result follows.

The converse of the above theorem need not be true as seen from the following example.

Example : 5.3 Consider the topological spaces X=Y={a, b, c} with topologies τ ={ φ ,X,{a},{b},{a,b}} and σ = { φ ,Y,{a, b}}. Here C(X, τ) = { φ ,X,{b, c},{a,c},{c}}, C(Y, σ) = { φ ,Y,{c}}and αcg -(Y, σ) = P(X). Let f be the identity map from X onto Y. Then
f is αcg -closed but not a closed map, since for the closed set {b, c} in (X, τ), f({b, c}) = {b, c} is not a closed set in Y.

Definition : 5.4 A map f : X →Y is called a αcg -open map if f(U) is αcg -open in Y for every open set U in X.

Theorem : 5.5 If f : X →Y is an open map, then it is αcg -open but not conversely.
Proof : Let f : X →Y be an open map. Let U be any open set in X. Then f(U) is an open set in Y. Then f(U) is αcg -open, since every open set is αcg -open. Therefore f is αcg -open.

The converse of the above theorem need not be true as seen from the following example.

Example : 5.6 Consider the topological spaces X=Y={a, b, c} with topologies τ ={ φ ,X,{a},{b},{a,b}} and σ = { φ ,Y,{a, b}}. Here αcgo -(Y, σ) = P(X). Then the identity function f: X →Y is αcg -open but not open, since for the open set {a} in (X, τ), f({a}) = {a} is αcg open but not open in (Y, σ). Therefore f is not an open map.

Theorem : 5.7 A map f : X →Y is αcg -closed iff for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a αcg -open set V of Y such that S $\subseteq$ V and $f^{-1}(V) \subseteq$ U.
Proof : Suppose f is αcg -closed. Let S be a subset of Y and U is an open set of X such that
$f^{-1}(S) \subseteq$ U. Then V = Y – f(X–U) is a αcg -open set containing S such that $f^{-1}(V) \subseteq$ U.

Conversely, suppose that F is a closed set in X. Then
f^{-1}(Y- f(F)) = X – F and X – F is open. By hypothesis, there is a αcg -open set V of Y such that Y- f(F) $\subseteq$ V and
$f^{-1}(V) \subseteq$ X – F. Therefore F $\subseteq$ X - f^{-1}(V). Hence
Y – V $\subseteq$ f(F) $\subseteq$ f(X - f^{-1}(V)) $\subseteq$ Y – V, which implies
f(F) = Y – V. Since Y – V is αcg -closed and thus f is αcg -closed map.

Theorem : 5.8 If f : X →Y is g-continuous and αcg -closed and A is a αcg -closed set of X, then f(A) is αcg -closed in Y.
Proof : Let f(A) $\subseteq$ O where O is an g-open set of Y. Since f is g-continuous, f^{-1}(O) is an g-open set containing A. Hence cl(A) $\subseteq$ f^{-1}(O) as A is a αcg -closed set. Since f is αcg -closed f(cl(A)) is a αcg -closed set contained in the g-open set O, which implies that cl(f(cl(A))) $\subseteq$ O and hence cl(f(A)) $\subseteq$ O. So f(A) is a αcg -closed set in Y.

Corollary : 5.9 If f : X →Y is continuous and closed and A is a αcg -closed set of X, then f(A) is αcg -closed in Y.
Proof : Since every continuous map is g-continuous and every closed map is αcg -closed, by above theorem the result follows.

Theorem : 5.10 If f : X →Y is a closed map and h : Y →Z is a αcg -closed then h : Y →Z is αcg -closed then h $\circ$ f : X →Z is a αcg -closed.
Proof : If f : X →Y is a closed map and h : Y →Z is a αcg -closed map. Let V be any closed set in X. Since f : X →Y is closed, f(V) is closed in Y and since h : Y →Z is αcg -closed, h(f(V)) is a αcg -closed set in Z. Therefore h $\circ$ f : X →Z is a αcg -closed map.

Theorem : 5.11 If f : X →Y is a αcg -closed and A is closed set in X. Then f_A : A→Y is αcg -closed.
Proof : Let V be closed set in A. Then V is closed in X. Therefore it is αcg closed in X. By Theorem 5.8, f(V) is αcg -closed. That is f_A(V) = f(V) is αcg -closed in . Therefore f_A : A→Y is αcg -closed.

The next theorem shows that normality is preserved under continuous and sc*g-closed maps.

Theorem : 5.12 If f : X →Y is a continuous, αcg -closed map from a normal space X onto a space Y, then Y is normal.
Proof : Let A, B be disjoint closed set Y. Then f^{-1}(A), f^{-1}(B) are disjoint closed sets of X. Since X is normal, there are disjoint open sets U, V in X such that $f^{-1}(A) \subseteq$ U and $f^{-1}(B) \subseteq$ V. By Theorem 5.7, and since f is αcg -closed, there are αcg -closed sets G, H in Y such that A $\subseteq$ G,
B $\subseteq$ H and $f^{-1}(G) \subseteq$ U and $f^{-1}(H) \subseteq$ V. Since U, V are disjoint, int(G) and int(H) are disjoint open sets. Since G is αcg -open, A is closed and A $\subseteq$ G $\Rightarrow$ A $\subseteq$ int(G). Similarly B $\subseteq$ int(H). Hence Y is normal.

6. CONCLUSION

In this paper we have introduced αcg-closed sets which is the weaker form of closed sets in topological spaces.And studied about the continuity and closed and open maps to the set. Some of the properties regarding these sets are discussed.

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