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Dominator Coloring on Star and Double Star Graph Families

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ABSTRACT
Dominator chromatic number of central, middle and total graphs of star and double star graph families are found in this paper. Also the relationship between dominator chromatic number and star chromatic number of the above graph families are obtained.

Key words
Central graph, middle graph, total graph and dominator coloring.

AMS Subject Classification
05C15, 05C69

1. PRELIMINARIES
In this section, we review the notions of central graph, middle graph, total graph and dominator coloring [1, 2, 3, 5].

Definition 1.1

Let G be a simple and undirected graph and let its vertex set and edge set be denoted by V(G) and E(G). The *central graph* of G, denoted by C(G) is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in C(G).

Definition 1.2

The *middle graph* M(G) of a graph G is defined as follows. The vertex set of M(G) is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of M(G) are adjacent in M(G) if either (i) x, y are in E(G) and x, y are adjacent in G or (ii) x is in V(G), y is in E(G) and x, y are incident in G. In other words, M(G) is obtained by subdividing each edge of G exactly once and joining all these newly added middle vertices of adjacent edges of G.

Definition 1.3

The *total graph* T(G) of a graph G is defined as a graph with vertex set $V(G) \cup E(G)$ and two vertices x, y of T(G) are adjacent in T(G) if either (i) x, y are in V(G) and x is adjacent to y in G or (ii) x, y are in E(G) and x, y are adjacent in G or (iii) x is in V(G), y is in E(G) and x, y are incident in G.

Definition 1.4

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A *Color class* is the set of vertices, having the same color. The color class corresponding to the color i is denoted by V_i.

Definition 1.5

A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

2. DOMINATOR COLORING ON CENTRAL GRAPH, MIDDLE GRAPH AND TOTAL GRAPH OF STAR GRAPH
In this section, dominator chromatic number of central graph, middle graph and total graph of the family of star graphs are obtained. Here it is assumed that the vertex set $V(K_{1, n})$ of the star graph $K_{1, n}$ is $\{v, v_1, v_2, \ldots, v_n\}$, where $v_1, v_2, \ldots, v_n$ are the pendent vertices of $K_{1, n}$ and the center vertex v is adjacent to v_i, $1 \le i \le n$.

Theorem 2.1

For star graph $K_{1, n}$, $n \ge 2$, $\chi_d[C(K_{1,n})] = n + 1$.

Proof

By the definition of central graph, $C(K_{1, n})$ is obtained by subdividing each edge vv_i of $K_{1, n}$ exactly once by the vertex c_i, $1 \le i \le n$ in $C(K_{1, n})$ and joining all the non-adjacent vertices v_iv_j of $K_{1, n}$, $1 \le i, j \le n$. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V' = \{c_1, c_2, \ldots, c_n\}$. Then $V(C(K_{1, n})) = V_1 \cup V' \cup \{v\}$, the set V′ is linearly independent and c_i is adjacent to v and v_i, i = 1, 2, …, n. Also observe that induced subgraph $< V_1 >$ is complete in $C(K_{1, n})$.

A procedure to obtain a dominator coloring of $C(K_{1, n})$ is as follows. Notice that $C(K_{1, n})$ contains a clique $< V_1 >$ of order n and hence n colors are required to color the vertices in V_1. Let v_i be colored by color i, $1 \le i \le n$. Since c_i is adjacent to v_i and non–adjacent with v_{i+1}, assign color i+1 to c_i, $1 \le i \le n-1$ and assign color 1 to c_n. As the vertex v is adjacent to all vertices c_i, $1 \le i \le n$, a new color (n+1) is assigned to v.

The above procedure guarantees a dominator coloring of $C(K_{1, n})$ as v_i dominates the color class i+1, $1 \le i \le n - 1$ and v_n dominates color class 1. Also vertices c_i,

$1 \le i \le n$ dominate the color class (n+1) and vertex v dominates itself. Hence $\chi_d[C(K_{1,n})] = n+1$.

The following example illustrates the procedure discussed in the above result.

Example 2.2

In figure 1, central graph of $K_{1,4}$ is depicted with a dominator coloring.

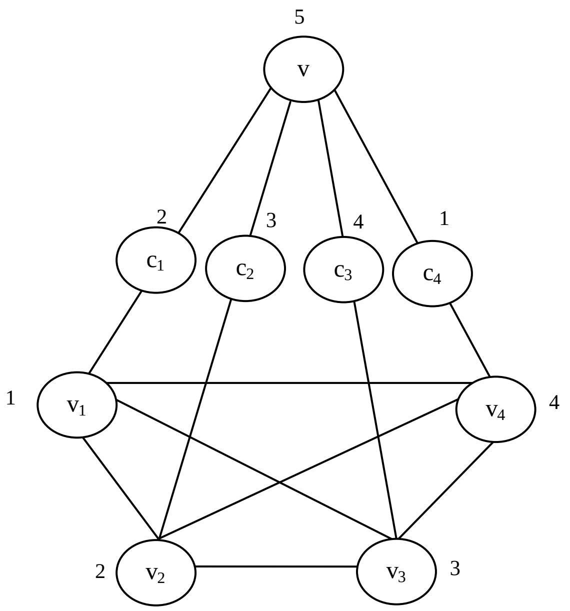

Figure 1

The color classes of $C(K_{1,4})$ are $V_1 = \{v_1, c_4\}$, $V_2 = \{v_2, c_1\}$, $V_3 = \{v_3, c_2\}$, $V_4 = \{v_4, c_3\}$ and $V_5 = \{v\}$. Therefore $\chi_d[C(K_{1,4})] = 5$.

Theorem 2.3

For star graph $K_{1,n}$, $n \ge 2$, $\chi_d[M(K_{1,n})] = n+1$.

Proof

It is clear from the definition of middle graph, $M(K_{1,n})$ is obtained by subdividing each edge vv_i of $K_{1,n}$ exactly once by the vertex c_i, $1 \le i \le n$ in $M(K_{1,n})$ and joining all these middle vertices c_i and c_j of adjacent edges of $K_{1,n}$, $1 \le i, j \le n$. Let $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V' = \{c_1, c_2, \dots, c_n\}$. Then $V(M(K_{1,n})) = V_1 \cup V' \cup \{v\}$and c_i is adjacent to v_i, $i = 1, 2, \dots, n$. Also notice that induced subgraph $< V' >$ is complete in $M(K_{1,n})$.

The following procedure gives a dominator coloring of $M(K_{1,n})$. As $M(K_{1,n})$ contains a clique of order n, minimum n colors are required to color the vertices $c_1, c_2, \dots, c_n$ of this clique. So the vertices $c_1, c_2, \dots, c_n$ are colored by the colors 1, 2, …, n. As the vertex v is adjacent to c_i, i = 1, 2, …, n, a new color (n+1) is assigned to v. Since v_i is independent and v_i and v are non-adjacent, the color (n+1) which is assigned to v can also be assign to v_i, $1 \le i \le n$.

In $M(K_{1,n})$, c_i dominates itself. Since v and v_i are adjacent to c_i, vertices v and v_i dominate the color class i, $1 \le i \le n$. So the given procedure gives a dominator coloring of $M(K_{1,n})$. Hence $\chi_d[M(K_{1,n})] = n+1$.

Theorem 2.4

For star graph $K_{1,n}$, $n \ge 2$, $\chi_d[T(K_{1,n})] = n+1$.

Proof

Observe from the definition of total graph that $T(K_{1,n})$ is obtained by subdividing each edge vv_i of $K_{1,n}$ exactly once by the vertex c_i, $1 \le i \le n$ in $T(K_{1,n})$ and joining all these middle vertices c_i and $c_j$of adjacent edges of $K_{1,n}$, $1 \le i, j \le n$ and also joining the adjacent vertices v and v_i of $K_{1,n}$, $1 \le i \le n$ in $T(K_{1,n})$. Let $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V' = \{c_1, c_2, \dots, c_n\}$. Then $V(T(K_{1,n})) = V_1 \cup V' \cup \{v\}$ and c_i is adjacent to v and v_i, i = 1, 2, …, n. The induced subgraph $< V' >$ is complete in $T(K_{1,n})$.

The procedure given in theorem 2.1 also gives a dominator coloring for $T(K_{1,n})$. It can be easily observed that all vertices in $T(K_{1,n})$ dominate color class n+1. Hence $\chi_d[T(K_{1,n})] = n+1$.

3. DOMINATOR COLORING ON CENTRAL GRAPH, MIDDLE GRAPH AND TOTAL GRAPH OF DOUBLE STAR GRAPH

In this section, we find the dominator chromatic number of central graph, middle graph and total graph of the double star graph families. Here it is assumed that the vertex set $V(K_{1,n,n})$ of double star $K_{1,n,n}$ is $\{v, v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$, where $u_1, u_2, \dots, u_n$ are the pendent vertices of $K_{1,n,n}$ and v_i is adjacent to u_i and v, $1 \le i \le n$.

Theorem 3.1

For double star graph $K_{1,n,n}$, $n \ge 2$, $\chi_d[C(K_{1,n,n})] = n+2$.

Proof

By the definition of central graph, $C(K_{1,n,n})$ is obtained by subdividing edge vv_i by c_i and v_iu_i by s_i, $1 \le i \le n$ of $K_{1,n,n}$ and joining all non-adjacent vertices of $K_{1,n,n}$. Let $V_1 = \{v_1, v_2, \dots, v_n\}$, $V_2 = \{u_1, u_2, \dots, u_n\}$, $V' = \{c_1, c_2, \dots, c_n\}$ and $V'' = \{s_1, s_2, \dots, s_n\}$. Then $V(C(K_{1,n,n})) = V_1 \cup V_2 \cup V' \cup V'' \cup \{v\}$, $\{c_1, c_2, \dots, c_n\}$ and $\{s_1, s_2, \dots, s_n\}$ are linearly independent and c_i is adjacent to v and v_i, $1 \le i \le n$. The induced subgraphs $< V_1 >$ and $< V_2 >$ are complete in $C(K_{1,n,n})$.

The following procedure gives a dominator coloring of $C(K_{1,n,n})$. As $C(K_{1,n,n})$ contains two cliques $< v_1, v_2, \dots, v_n >$ and $< u_1, u_2, \dots, u_n >$ each of order n and v_i and u_i, $1 \le i \le n$ are non- adjacent, n colors are sufficient to color the vertices $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$. So vertices v_i and u_i, are colored by color i, $1 \le i \le n$. Since c_i and s_i are non- adjacent, assign color (n+1) to c_i and s_i, $1 \le i \le n$. As vertex v is adjacent to c_i and u_i, $1 \le i \le n$, a new color (n+2) is assigned to v.

In $C(K_{1,n,n})$, v_i dominates the color class $i+1$, $1 \le i \le n-1$ and v_n dominates color class 1. Since s_i is adjacent to v_i and u_i, s_i dominates the color class i, $1 \le i \le n$. As c_i and u_i are adjacent to v, c_i and u_i dominate the color class $(n+2)$, $1 \le i \le n$. The vertex v dominates itself. So the given procedure gives a dominator coloring of $C(K_{1,n,n})$ and hence $\chi_d[C(K_{1,n,n})] = n+2$.

The following example illustrates the procedure discussed in the above result.

Example 3.2

In figure 2, central graph of $K_{1,3,3}$ is depicted with a dominator coloring.

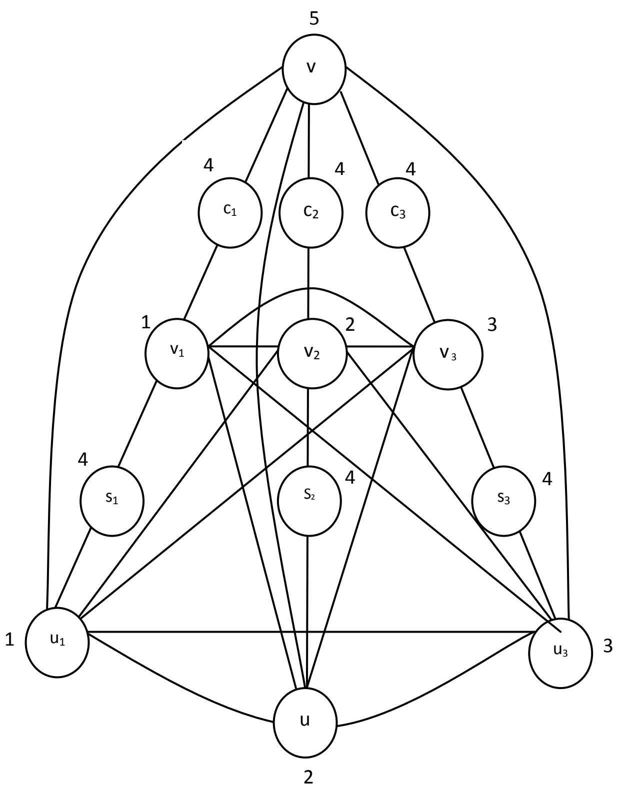

Figure 2

The color classes of $C(K_{1,3,3})$ are $V_1 = \{v_1, u_1\}$, $V_2 = \{v_2, u_2\}$, $V_3 = \{v_3, u_3\}$, $V_4 = \{c_1, c_2, c_3, s_1, s_2, s_3\}$ and $V_5 = \{v\}$. Therefore $\chi_d[C(K_{1,3,3})] = 5$.

Theorem 3.3

For double star graph $K_{1,n,n}$, $n \ge 2$, $\chi_d[M(K_{1,n,n})] = 2n+1$.

Proof

Observe that $M(K_{1,n,n})$ is obtained from $K_{1,n,n}$ by subdividing edge vv_i by c_i and v_iu_i by s_i, $1 \le i \le n$ and joining all these middle vertices of adjacent edges of $K_{1,n,n}$. Let $V_1 = \{v_1, v_2, \dots, v_n\}$, $V_2 = \{u_1, u_2, \dots, u_n\}$, $V' = \{c_1, c_2, \dots, c_n\}$ and $V'' = \{s_1, s_2, \dots, s_n\}$. Then $V(M(K_{1,n,n})) = V_1 \cup V_2 \cup V' \cup V'' \cup \{v\}$, $\{s_1, s_2, \dots, s_n\}$ is linearly independent and c_i is adjacent to v and v_i, $1 \le i \le n$. The induced subgraph $< V' >$ is complete in $M(K_{1,n,n})$.

A procedure to obtain a dominator coloring of $M(K_{1,n,n})$ is given below. As $M(K_{1,n,n})$ contains a clique $< V' >$ of order n, we require n colors to color the vertices of this clique. So vertex c_i is colored by color i, $1 \le i \le n$. Since the vertex v is adjacent to clique $< V' >$, a new color $(n+1)$ is used to color v. Since v_i, u_i and v are non-adjacent, assign the color $(n+1)$ to v_i and u_i, $1 \le i \le n$. As s_i is adjacent to v_i, u_i and c_i, assign color $(n+i+1)$ to s_i, $1 \le i \le n$.

In $M(K_{1,n,n})$, c_i dominates itself, $1 \le i \le n$ and v dominates all color classes 1 to n. Since s_i is adjacent to v_i, u_i and c_i, the vertices v_i and u_i dominate the color class $(n+i+1)$ and s_i dominates itself, $1 \le i \le n$. So the given procedure gives a dominator coloring of $M(K_{1,n,n})$ and hence $\chi_d[M(K_{1,n,n})] = 2n+1$.

Theorem 3.4

For double star graph $K_{1,n,n}$, $n \ge 2$, $\chi_d[T(K_{1,n,n})] = 2n+1$.

Proof

It is clear from the definition of total graph, $T(K_{1,n,n})$ is obtained from $K_{1,n,n}$ by subdividing edge vv_i by c_i and v_iu_i by s_i, $1 \le i \le n$, joining all these middle vertices of adjacent edges of $K_{1,n,n}$ and also joining all adjacent vertices of $K_{1,n,n}$. Let $V_1 = \{v_1, v_2, \dots, v_n\}$, $V_2 = \{u_1, u_2, \dots, u_n\}$, $V' = \{c_1, c_2, \dots, c_n\}$ and $V'' = \{s_1, s_2, \dots, s_n\}$. Then $V(T(K_{1,n,n})) = V_1 \cup V_2 \cup V' \cup V'' \cup \{v\}$, $\{s_1, s_2, \dots, s_n\}$ is linearly independent and c_i is adjacent to v and v_i, $1 \le i \le n$. The induced subgraph $< V' >$ is complete in $T(K_{1,n,n})$.

A following procedure gives a dominator coloring of $T(K_{1,n,n})$. As $T(K_{1,n,n})$ contains a clique $< V' >$ of order n, we require n colors to color the vertices of this clique. That is, vertex c_i is colored by color i, $1 \le i \le n$. Since v_i is adjacent to c_i, s_i, u_i and v, assign color $i+1$ to v_i, $1 \le i \le n-1$ and assign color 1 to v_n. As the vertex v is adjacent to c_i, $1 \le i \le n$, assign a new color $(n+1)$ to v. Since v and u_i are non-adjacent, u_i is colored by the same color $(n+1)$, $1 \le i \le n$. As s_i is adjacent to v_i and u_i, assign the colors $(n+i+1)$ to s_i, $1 \le i \le n$.

In $T(K_{1,n,n})$, c_i dominates itself, $1 \le i \le n$ and v dominates all color classes 1 to n. Since s_i is adjacent to v_i and u_i, the vertices v_i and u_i dominate the color class $(n+i+1)$ and s_i dominates itself. So the given procedure gives a dominator coloring of $T(K_{1,n,n})$ and hence $\chi_d[T(K_{1,n,n})] = 2n+1$.

By combining the observations of [4] and theorems 3.1, 3.3 and 3.4, we have the following result.

Result 3.5

(i) $\chi_d[C(K_{1,n,n})] = X_s[C(K_{1,n,n})] - (n-1)$

(ii) $\chi_d[M(K_{1,n,n})] = X_s[M(K_{1,n,n})] + n$

(iii) $\chi_d[T(K_{1,n,n})] = X_s[T(K_{1,n,n})] + n$.

4. CONCLUSION

In this paper, we obtained the dominator chromatic number of star and double star graphs and compared these parameters with star chromatic number of the corresponding graph families. This paper can further be extended by identifying graph families of graphs for which these two chromatic numbers are equal.

5. ACKNOWLEDGMENTS

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