Connected Network Dominating Set of an Interval Graph

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ABSTRACT
Connected dominating sets are useful in the computation of routing for mobile ad-hoc networks. A connected dominating set is used as a backbone for communications, and nodes that are not in this set communicate by passing messages through neighbors that are in the set. Recent advances in technology have made possible the creation of Wireless Sensor Networks. Although there is no physical backbone infrastructure, a virtual backbone can be formed by constructing a Connected Dominating Set (CDS). In this paper we present an algorithm for finding minimal connected network dominating set(MCNDS) of an interval graph.

Key Words
Interval family, Interval graph, Dominating set, Connected dominating set, Network.

1. INTRODUCTION
A subset D of V is said to be dominating set[1],[2] if every vertex in $<V-D>$ is adjacent to some vertex in D.The minimum cardinality of a dominating set is called dominating number. It is denoted by γ(G). As Hedetniemi & Laskar[3] note, the domination problem was studied from the 1950s onwards, but the rate of research on domination significantly increased in the mid-1970s A dominating set D is said to be connected dominating set if the induced sub graph $<D>$ is connected.. A minimum connected dominating set of a graph is a connected dominating set with the smallest possible cardinality among all connected dominating set[4],[5] of graph. The connected domination number of graph G is the number of vertices in the minimum connected dominating set of graph G[6].It is denoted by γ_c (G). Let $G(V,E)$ be a graph. The closed neighbourhood of a vertex v in G is defined as v and the set of vertices that are adjacent to v in G. The closed neighbourhood is denoted by $N[v]$. A set S of vertices in G is called a neighbourhood Set[7] of G if $G=\bigcup_{v\in S}\langle N[v]\rangle$, where $\langle N[v]\rangle$ is the sub graph of G induced by $N[v]$. A subset D of V is called a dominating set of V if every vertex in $V-D$ is adjacent to some vertex in D. A connected dominating set[8],[9] C is a dominating of G which induces a connected sub graph of G.

Let $I=\{I_1,I_2,...,I_n\}$ be an interval family where each I_i is an interval on the real line and $I_i=[a_i,b_i]$ for $i=1,2,3,....,n$. Here a_i is called the left endpoint and b_i the right endpoint of I_i. Without loss of generality, we assume that all endpoints of the intervals in I are distinct numbers between 1 and $2n$. Two intervals i and j are said to intersect each other if they have non-empty intersection. A graph $G(V,E)$ is called an interval graph if there is a one-to-one correspondence between V and I such that two vertices of G are joined by an edge in E if and only if their corresponding intervals in I intersect. That is if $i=[a_i,b_i]$ and $j=[a_j,b_j]$, then i and j intersect means either $a_j<b_i$ or $a_i<b_j$

For each interval i let $nbd[i]$ denote the set of intervals that intersect i (including i). Let $\min(i)$ denote the smallest interval and $\max(i)$ the largest interval in $nbd[i]$.

Let us now define the non intersecting interval $NI(i)$ of the interval i as below:

$NI(i)=j$, if $b_i<a_j$ and there do not exist an interval k such that $b_i<a_k<a_j$. If there is no such j, then define $NI(i)=null$.

Define $Next(i)=\min(\{nbd[NI(i)]\}\setminus\{nbd[\min(i)]\})$ this concept was introduced by [2] We confine our discussion to connected graphs only.

First we augment I with two dummy intervals say, I_0 and I_{n+1}, where $I_0=[a_0,b_0]$ and $I_{n+1}=[a_{n+1},b_{n+1}]$ such that $b_0<\min_{1\le k\le n}\{a_k\}$ and $a_{n+1}>\max_{1\le k\le n}\{b_k\}$

Let $I_d=I\cup\{I_0,I_{n+1}\}$. We assume that the intervals in I_d are indexed by increasing order of their right end points namely $b_0<b_1<........<b_{n+1}$. In what follows we construct a directed network and show that the vertices in any shortest directed path in it correspond to a CNDS[3] of G.

Now a directed network $D(N,L)$ is constructed as follows: The nodes in N correspond to the intervals in I_1 which are not properly contained within other intervals. The lines in L are partitioned into two disjoint sets L_1 and L_2 are defined below. For $j\in D$, there is a directed line (I_0,j) between I_0 and j that belongs to L_1 if and only if there is no interval I_h such that $b_0<a_h<b_h<a_j$. Similarly there is a directed line (j,I_{n+1}) between j and I_{n+1} that belongs to L_2 if and only if there is no interval I_h such that $b_j<a_h<b_h<a_{n+1}$. This gives the scope to join the intervals I_0 and I_{n+1}.
to other intervals in I and it is obvious that all such joined directed lines belong to L_1. Next for i, $j\in D$, there is a directed line (i,j) between i and j that belongs to L_2 if and only if $j=Next(i)$.

2. ALGORITHM

Input: Interval family $I = \{1,2,....,n\}$.

Output: Minimal Connected Network dominating Set of an interval graph from a directed network D.

Step 1 : *for* i =1 to n

{

$p = NI(i)$

$q = \min(i)$

$R = nbd[p]$

$S = nbd[q]$

$Next(i) = \min(\{R\}\setminus\{S\})$

If $Next(i) = null$ then goto step1

else

Join i to $Next(i)$ and goto step 1

}

Step 2 : *for* j =2 to n

{

If I_1 is intersect to I_j contained in I_j.

Join I_0 to I_j.

If I_{n+1} is intersect to I_j contained in I_j.

Join I_{n+1} to I_j.

}

Step 3 : Find paths $P_1, P_2,...., P_k$ for some k from node I_0 to I_{n+1}.

Step 4 : *for* j =1 to k

{

If Nodes of P_j are connected in G

$MCNDS = \{nodes \text{ of } P_j\}$

}

Step 5 : End

3. ILLUSTRATIONS

Neighbours of Interval graph corresponding Interval family I(see Figure.1).

Nbd[1] = {1,2,3}, Nbd[2] = {1,2,3,4}
Nbd[3] = {1,2,3,4}, Nbd[4] = {2,3,4,5,6}
Nbd[5] = {4,5,6,7}, Nbd[6] = {4,5,6,7,9}
Nbd[7] = {5,6,7,8,9}, Nbd[8] = {7,8,9,10}
Nbd[9] = {6,7,8,9,10}, Nbd[10] = {8,9,10}

min(1) = 1, min(2) = 1, min(3) = 1, min(4) = 2
min(5) = 4, min(6) = 4, min(7) = 5, min(8) = 7
min(9) = 6, min(10) = 8

NI(1) = 4 , NI(2) = 5, NI(3) = 5,
NI(4) = 7, NI(5) = 8, NI(6) = 8
NI(7) = 10, NI(8) = null, NI(9) = null
NI(10) = null

$Next(i) = \min(\{nbd[NI(i)]\}\setminus\{nbd[\min(i)]\})$

$Next(1) = \min(\{nbd[NI(1)]\}\setminus\{nbd[\min(1)]\})$

$= \min(\{nbd[4]\}\setminus\{nbd[1]\})$

$= \min(\{2,3,4,5,6\}\setminus\{1,2,3\})$ =4

Similarly

$Next(2) = \min(\{nbd[NI(2)]\}\setminus\{nbd[\min(2)]\})$
$= 4.$

$Next(3) = \min(\{nbd[NI(3)]\}\setminus\{nbd[\min(3)]\})$
$= 4.$

$Next(4) = \min(\{nbd[NI(4)]\}\setminus\{nbd[\min(4)]\})$
$= 5.$

$Next(5) = \min(\{nbd[NI(5)]\}\setminus\{nbd[\min(5)]\})$
$= 7.$

$Next(6) = \min(\{nbd[NI(6)]\}\setminus\{nbd[\min(6)]\})$
$= 7.$

$Next(7) = \min(\{nbd[NI(7)]\}\setminus\{nbd[\min(7)]\})$
$= 8.$

$Next(8) = \min(\{nbd[NI(8)]\}\setminus\{nbd[\min(8)]\})$
$= null.$

$Next(9) = \min(\{nbd[NI(9)]\}\setminus\{nbd[\min(9)]\})$
$= null.$

$Next(10) = \min(\{nbd[NI(10)]\}\setminus\{nbd[\min(10)]\})$ Now
$= null.$

the dummy intervals I_0 and I_{n+1} are augmented to I as shown in Figure 2.

4. DIRECTED NETWORKS

$D(N, L)$

$N = \{0,1,2,3,4,5,6,7,8,9,10,11\}$.

$L = L_1 \cup L_2$

From Figure 3. $0-3-4-5-7-8-11$.
$0-2-4-5-7-8-11$.
$0-1-4-5-7-8-11$.

$CNDS = \{3,4,,5,7,8\},\{2,4,5,7,8\}$

5. MAIN RESULTS

Lemma1: Let $I = \{I_1, I_2,...., I_n\}$ be an interval family. If i and k are any two intervals which are intersecting and j is such that $i < j < k$, then j intersects k.

Proof: Given that s is an interval family .Since the intervals are labeled in increasing order of their right end points, it is with is easy to see that when $i < j < k$, then $b_i < b_j < b_k$. Now i intersects k implies that $a_k < b_i$, where a_k is the left end point and b_i is the rightend point. Therefore $a_k < b_i < b_j < b_k$ which implies that j also intersects k.

Lemma2: If the directed line $(0, j) \in L_1$ where j is any interval of I, then the intervals between 0 and j belongs to $nbd[j]$.

Proof: Let $I = \{I_1, I_2,...., I_n\}$ be an interval family. Suppose that $(0, j) \in L_1$.By the definition of lines in L_1 it follows that there is no interval I_h such that $b_0 < a_h < b_h < a_j$. So any interval between I_0 and I_j must intersect with I_j. Therefore the intervals between 0 and j belongs to $nbd[j]$.

Lemma3: If the directed line $(j, n+1) \in L_1$ where j is any interval of I then the intervals between j and n+1 are connected by j.

Proof: The proof of the above lemma follows on similar lines to that of lemma2.

Thus it is clear by lemma2 &3 that if there is a directed line $(i,j)\in L_1$ then the intervals between I_i and I_j are adjacent with I_i or I_j.

Lemma4: If the directed line $(i,j)\in L_2$ then the intervals between i and j are dominated by i or j but not both.

Proof: Let $(i,j)\in L_2$ then $j = Next(i)$.The lemma follows immediately if either h intersect i or j. If $h = NI(i)$,then by the definition of j clearly h intersects j. So $h \neq NI(i)$.Then two cases will arise .

Case1: Suppose h intersects $NI(i)$.Again this implies by the definition of $Next(i) = \min(\{nbd[NI(i)]\} \setminus \{nbd[\min(i)]\})$.
That h intersects j.

Case2: Suppose h doesn't intersect $NI(i)$. Then $NI(i) < h$. By our assumption $i < h < j$.Therefore $NI(i) < h < j$. By the definition of j, $NI(i)$ and j intersects. Hence by lemma1 h and j intersects. Thus for all cases either h intersects i or j which implies that the intervals between i and j are dominated by i or j. We now show that if h is any interval between i and j then h is not dominated by both i and j. Since
$Next(i) = \min(\{nbd[NI(i)]\} \setminus \{nbd[\min(i)]\}),\ \min(i)$
does not intersect $Next(i) = j$.
So the intervals between i and $\min(i)$ does not intersect j and the intervals between $\min(i)$ and j does not interval i. Hence there is no interval h between i and j that intersect both i and j.

Lemma5: If the connected network dominating set(CNDS) denote the set of vertices in the shortest path between the nodes 0 and n+1 in the directed network $D(N,L)$ then there will not be any interval in I such that it intersects all the intervals in CNDS.

Proof: Suppose there is an interval k in I such that k is dominated by all the intervals in connected network dominating set. Let $CNDS = \{i_1, i_2, ..., i_n\}$ Then we have either
$i_1 < i_2 < .... < i_n < k$ (or) $i_1 < i_2 < .... < i_{n-1} < k < i_n$
Suppose $i_1 < i_2 < .... < i_n < k$
Then $\min(i_1) = k$ and

$$nbd[\min(i_1)] = nbd[k]$$
$$= \{i_1, i_2, ..., i_n, k\}$$

$V(G) = \bigcup_{I_i \in S} N[I_i]$,
it follows that

Now $nbd[NI(i_1)] = \{NI(i_1), ..., i\}$ and hence $\{nbd[NI(i_1)]\} \setminus \{nbd[NI(i_1)]\} = null$. That is $Next(i_1) = i_2 = null$, a contradiction. Similar is the case if $i_1 < i_2 < .... < i_{n-1} < k < i_n$
Thus there is no interval k in I such that k intersects all the intervals in CNDS.

Theorem1: The vertices in a path between nodes 0 and n+1 in $D(N,L)$ corresponding to a connected network dominating set in G having interval properly contained with in any interval of the given interval graph G of I.

Proof: Let P be a path from node 0 to n+1 in $D(N,L)$
Define
$S = \{I_i : node \text{ i appears in P }, i \neq 0, i \neq n+1\}$
For each directed line (i,j) in P by lemmas 2,3&4 it follows that all intermediate intervals $I_{i+1}, I_{i+2}, ..., I_{j-1}$ between I_i and I_j are belongs to $N[I_i] \cup N[I_j]$. Hence , all the intermediate intervals between the intervals in S belongs to $\bigcup_{I_i, I_j \in S} G[N[I_i] \cup N[I_j]]$.

Since the intervals in S correspond to nodes in the path P between I_0 and I_{n+1},the interval between I_0 and the first interval in S, as well as the interval between the last interval in S and I_{n+1},also belongs $\bigcup_{I_i \in S} G[N[I_i]]$. Thus, all nodes in G are adjacent to nodes in S. That is
$V(G) = \bigcup_{I_i \in S} N[I_i]$.

But the subgraph H by the vertex set $\{I_i, I_{i+1}, ..., I_j\}$ is a subgraph of the induced graph $G[N[I_i] \cup N[I_j]]$, where (I_i, I_j) is any line in CNDS. Therefore
$G[I_i, ..., I_j] \subseteq G[N[I_i] \cup N[I_j]]$,where $I_i, I_j \in S$. Therefore
$\bigcup_{I_i, I_j \in S} G[\{I_i, ..., I_j\}] \subseteq G[N[I_i] \cup N[I_j]]$
Since
$G = \bigcup_{I_i \in S} G[N[I_i]]$.

Thus, S is neighbourhood set of G[I]. By lemma5, the nodes in S are non adjacent.
Therefore, S forms a connected network dominating set of G[I].Since P is a path ,it follows that S is Minimum CNDS of G[I].

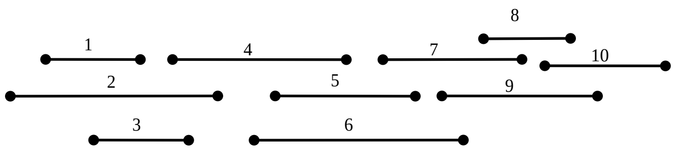

Figure 1Interval family I

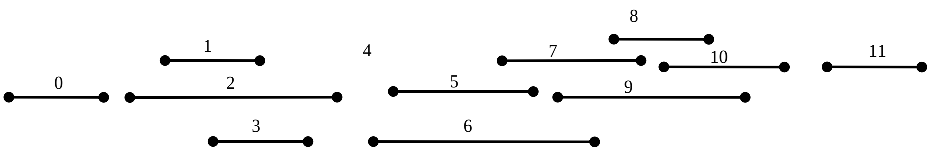

Figure 2The dummy intervals I_0 and I_{n+1} are augmented to I.

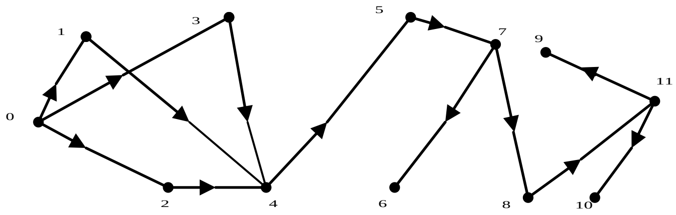

Figure 3.Directed Network D (N, L)

6 . CONCLUSIONS

In this paper we discussed at the connected network dominating sets in wireless communication networks. The interval graphs are rich in combinatorial structures and have found applications in several disciplines such as wireless communication networks which are wireless sensor networks and wireless ad-hoc networks , computer science and particularly useful in cyclic scheduling and computer storage allocating problems. We then extend the results to trace put a specific types of an interval graphs. We presented an algorithm to identify an interval graphs having a connected network
dominating set.

7. REFERENCES

[1] Teresa W.; Hedetniemi, Stephen; Slater, Peter (1998a), *Fundamentals of Domination in Graphs*, Marcel Dekke.

[2] Haynes, Teresa W.; Hedetniemi, Stephen; Slater, Peter (1998b), *Domination in Graphs: Advanced Topics*, Marcel Dekker.

[3] Hedetniemi, S. T.; Laskar, R. C., "Bibliography on domination in graphs and some basic definitions of domination parameters", Discrete Mathematics **86** (1–3): 257–277, 1990.

[4] Sampathkumar, E.; Walikar, HB, "The connected domination number of a graph", *J. Math. Phys. Sci 13 (6):*

607-613, 1979.

[5] K.M. Alzoubi, P.J.Wan, and O.Frieder. Message-optimal connected-dominating- set construction for routing in mobile ad hoc networks. In MOBIHOC' 02,pages 157-164, 2002.

[6] C.Adjih,P.Jacquet,and L.Viennot " Computing connected Dominated sets with multipoint relays," Technical Report 4597, INRIA-Rapport de recherché, Oct.2002.

[7] Maheswari.B, Lakshmi Naidu.Y, Naga muni Reddy.L, and A.Sudhakaraiah, "A polynomial time algorithm for finding a minimum independent neighbourhood set of an interval graph", Graph theory Notes of Newyork XLVI, 9-12,2004.

[8] G.Ramalingan and C.Pandu Rangan ; "Unified approach to domination problems on interval graphs". Inform.Process Lett., 27, 271-274,1989.

[9] Guha, S.; Khuller, S, "Approximation algorithms for connected dominating sets", *Algorithmica* **20** (4): 374–387, 1998.