Robust fault detection for Takagi-Sugeno discrete models: Application for a three-tank system

H. Ghorbel, M. Souissi, M. Chaabane
Laboratory of Sciences and Techniques of Automatic control & computer engineering, National School of Engineering of Sfax, University of Sfax, PB 1173, 3038 Sfax, Tunisia.

F. Tadeo

ABSTRACT
In this paper, we present a fuzzy observer based on Takagi-Sugeno (TS) models, to estimate simultaneously the system state and the sensors faults of discrete time nonlinear systems. The method uses the technique of descriptor systems, by considering the sensor faults as auxiliary states variables. Then the TS system is written as follows:

The TS model described by fuzzy IF-THEN rules. The 6th rule of the model is of the following form:

Rule i

IF \( z_i(k) \) is \( M_y \) and \( \ldots \) \( z_q(k) \) is \( M_y \) THEN

\[
\begin{align*}
    x(k+1) &= A_i x(k) + B_i u(k) \\
    y(k) &= C x(k)
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( u(k) \in \mathbb{R}^m \) is the input vector and \( y(k) \in \mathbb{R}^p \) is the measurable output vector. \( A, B, \) and \( C \) are matrices with appropriate dimension. \( z_i(z_i(k) \ldots z_q(k)) \) are the premise variables, \( M_{y_i} \ldots M_y \) are the fuzzy sets and \( r \) is the number of rules.

Then the TS system is written as follows:

\[
\begin{align*}
    x(k+1) &= \sum_{i=1}^{r} \mu_i(z_i)(A_i x(k) + B_i u(k)) \\
    y(k) &= C x(k)
\end{align*}
\]

where
\[
\mu_i(\xi(t)) = \frac{w_i(\xi(t))}{\sum_{i=1}^{r} w_i(\xi(t))}, \quad w_i(\xi(t)) = \prod_{i=1}^{q} M_{ij}(\xi(t))
\]  

Hence, \( \mu_i(\xi) \) satisfies:

\[
\sum_{i=1}^{r} \mu_i(\xi) = 1 \quad \text{and} \quad \mu_i(\xi) \geq 0 \quad \text{for} \quad i = 1, \ldots, r
\]  

In this paper, we considered TS models in discrete time with sensor faults. Then, the system (2) is rewritten in the following form:

\[
\begin{align*}
\dot{x}(k+1) &= \sum_{i=1}^{r} \mu_i(\xi) (A_i x(k) + B_i u(k)) \\
y(k) &= C_i x(k) + f_i(k)
\end{align*}
\]  

where \( f_i(k) \) represents the additive sensor fault and \( D_i \) is a matrix of appropriate dimensions.

In order to estimate the state and the sensor faults, we considered the faults as an auxiliary state of the augmented system.

An augmented system descriptor is then constructed as follows:

\[
\begin{align*}
\dot{z}(k+1) &= \sum_{i=1}^{r} \mu_i(\xi) (A_i z(k) + B_i u(k)) + D_i x_i(k) \\
y(k) &= C z(k) = C_0 z(k) + x_i(k)
\end{align*}
\]  

where:

\[
A_i = \begin{bmatrix} A & 0 \\ 0 & -I_n \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_i = (C \ 0), \quad \bar{C} = (C \ I_p)
\]  

In this study, we considered the following observer structure:

\[
\begin{align*}
E z(k+1) &= \sum_{i=1}^{r} \mu_i(\xi) (N_i z(k) + \bar{B}_i u(k)) \\
\hat{x}(k) &= z(k) + M y(k)
\end{align*}
\]  

where \( z(k) \in \mathbb{R}^{n+p} \) is an auxiliary state vector, \( \hat{x}(k) \in \mathbb{R}^{n+p} \) is the estimate of \( \pi(k) \in \mathbb{R}^{n+p} \). \( E, \ N_i \in \mathbb{R}^{(n+p)x(n+p)} \) and \( M \in \mathbb{R}^{n+ppx(n+p)} \) are the design parameters of the observer.

### 2.2 Residual generation

The error estimation and the residual signal are defined as:

\[
\begin{align*}
e(k) &= \pi(k) - \hat{\pi}(k) \\
r(k) &= V(y(k) - \hat{y}(k))
\end{align*}
\]  

where \( V \) is the residual weighting matrix.

**Definition:**

Given system (2) and a scalar \( \beta > 0 \), the observer (7) is called a \( H_{\beta} \) fault detection observer if it is asymptotically stable and the following inequality is satisfied:

\[
r^T(k) r(k) \geq \beta^2 f_i^T(k) f_i(k)
\]  

The objective is to design an allowable observer (7) to maximize \( \beta \), i.e., a sensitive fault observer. In the following, we determine the error dynamics equation. From the equations (6) and (7), we considered the following system:

\[
(\bar{E} + E \bar{C}) \bar{\pi}(k+1) - E \hat{x}(k+1) = \sum_{i=1}^{r} \mu_i(\xi)
\]  

In order to determine the observer gains, we consider the following assumption:

\[
N_i = \bar{A}_i + N_i M C_0
\]  

\[D = -N_i M\]

\[
E = \bar{E} + E M \bar{C}
\]  

The error dynamics can be written as follows:

\[
E \pi(k+1) = \sum_{i=1}^{r} \mu_i(\xi) F_i \pi(k) \quad \text{for} \quad i = 1, \ldots, r
\]  

We can easily show that the constraints (12) are checked with the following choice:

\[
\begin{align*}
N_i &= \begin{bmatrix} A_i & 0 \\ -C & -I_n \end{bmatrix} \\
M &= \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\
E &= \begin{bmatrix} I_n \\ QC \ Q \end{bmatrix}
\end{align*}
\]  

where \( Q \in \mathbb{R}^{pp} \) is a full-rank matrix, that is a parameter to determine. By taking account of (14)-(16), the error dynamics becomes:

\[
\bar{\pi}(k+1) = \sum_{i=1}^{r} \mu_i(\xi) S_i \bar{\pi}(k) \quad \text{for} \quad i = 1, \ldots, r
\]  

where:

\[
S_i = E^{-1} N_i = \begin{bmatrix} I_n \\ QC \ Q \end{bmatrix}^{-1} \begin{bmatrix} A_i & 0 \\ -C & -I_n \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA -Q^{-1} C & -Q^{-1} \end{bmatrix}
\]

**Remark:**

Therefore, the problem of the observer synthesis comes down to the determination of the matrix \( R \) and the residual weighting matrix \( V \) such that simultaneously:

- The observation error \( e(k) \) aims asymptotically towards zero;
- The residual \( r(k) \) provides good fault sensitivity.

We can write in this case:

\[
r(k) = V(C \bar{\pi}(k) + D_i f_i(k))
\]
3. SYNTHESIS OF THE ROBUST FAULT OBSERVER

In this section, we consider the robust residual problem $r(k)$ via the fault $f(k)$. To attain the objective, the $H_{\infty}$ performance is used as measurement of the worst case fault sensitivity of the residual generator on the presence of a fault.

**Theorem:**

The system (17) and (18) is asymptotically stable and guarantee the performance (10), if there exist matrices $P_i > 0$, matrices $Z_i$, $Z_s$ and $V$ and a scalar $\beta > 0$, such that the following LMI holds:

$$\begin{bmatrix}
\Lambda_i + A_i^T P A_i + A_i^T C_i^T P_i C_i + A_i^T Z_i C_i + Z_i^T A_i - P_i \\
2G_i(V_i V_k^T) \quad A_i^T C_i^T Z_2^T \\
* & -P_2 \\
* & * & \beta^2 I + Z_2^T D_2^T V_z^T D_2 \\
* & * & * & -I \\
* & * & * & * & -P_2 \\
\end{bmatrix} < 0$$

(20)

where:

$$\Lambda_i = A_i^T P A_i + A_i^T C_i^T P_i C_i + A_i^T Z_i C_i + Z_i^T A_i - P_i$$

$$G_i(V_i V_k^T) = (V_i^T)^T V_k^T - (V_k^T)^T V_i - C_i^T V_i V_k$$

$$G_2(V_i V_k^T) = (V_i^T)^T V_k^T - (V_k^T)^T V_i - D_2^T V_i V_k$$

The observer (7) is defined by (14)-(16) with $Q = (P_2^{-1} Z_2)^{-1}$.

**Proof:**

The candidate Lyapunov function is defined as:

$$V(k) = \tilde{z}_d^T(k) P \tilde{z}_d(k), \ P > 0$$

(23)

$$\Delta V = V(k+1) - V(k)$$

$$= \tilde{z}_d^T(k+1) P \tilde{z}_d(k+1) - \tilde{z}_d^T(k) P \tilde{z}_d(k)$$

Thus,

$$\tilde{z}(k+1) = \sum_{i=1}^{R} \mu_i(\xi) S_i \tilde{z}(k)$$

Then, $\Delta V = \sum_{i=1}^{R} \mu_i(\xi) \tilde{z}_d^T(S_i^T P S_i - P) \tilde{z}_d(k)$

$$J_- = r^T(k) (r(k) - \beta^2 f_1^T f_1) (k)$$

$$= r^T(k) (r(k) - \beta^2 f_1^T f_1) (k) - \Delta V(\xi) + \Delta V(\xi)$$

$$= ((G_2^T + D_2 f_1)^T V_i (G_2^T + D_2 f_1) - \beta^2 f_1^T f_1 - \sum_{i=1}^{\infty} \mu_i(\xi) \tilde{z}_d^T(S_i^T P S_i - P) \tilde{z}_d(k) + \Delta V(\xi)$$

$$J_- = \sum_{i=1}^{\infty} \mu_i(\xi) (S_i^T P S_i - P) \tilde{z}_d(k) + \Delta V(\xi)$$

With $\gamma_i = -(S_i^T P S_i - P) + C_0^T V_i C_1 + C_0^T V_i^T D_2 + \beta^2 I - D_2^T V_i V_k^T D_2$.

Consequently, if $\gamma_i \geq 0$, we can guarantee $J_- \geq 0$, that is,

$$\begin{bmatrix}
S_i^T P S_i - P - C_0^T V_i C_1 & C_0^T V_i^T D_2 \\
* & \beta^2 I - D_2^T V_i V_k^T D_2
\end{bmatrix} \leq 0$$

(25)

Replacing $S_i = \begin{bmatrix} A_i & 0 \\ -C_i - R_i^T C_i & -R_i \end{bmatrix}$ and supposing that $P = \text{diag}(P_1, P_2)$, we obtain:

$$\begin{bmatrix}
\Lambda_i - C_i^T V_i^T C_1 & \Omega_i & C_i^T V_i^T D_2 \\
* & R_i^T P_2 R_i^{-1} - P_2 & 0 \\
* & * & \beta^2 I - D_2^T V_i V_k^T D_2
\end{bmatrix} \leq 0$$

(26)

where

$$\Lambda_i = A_i^T P A_i + A_i^T C_i^T P_i C_i + A_i^T Z_i C_i + Z_i^T A_i - P_i$$

$$\Omega_i = A_i^T C_i P_i R_i^{-1} + C_i^T R_i^{-1} C_i$$

Applying the Schur complement theorem to the LMI (26) and by the following change of variables

$$V_i^k = V_i^{k-1} D_2$$ and $V_k^k = V_k^{k-1} C_i$, we obtain:

$$\begin{bmatrix}
\Lambda_i + A_i^T P A_i + A_i^T C_i^T P_i C_i + A_i^T Z_i C_i + Z_i^T A_i - P_i \\
2G_i(V_i V_k^T) \quad A_i^T C_i^T Z_2^T \\
* & -P_2 \\
* & * & \beta^2 I + Z_2^T D_2^T V_z^T D_2 \\
* & * & * & -I \\
* & * & * & * & -P_2 \\
\end{bmatrix} \leq 0$$

(27)

This LMI (27) can also be rewritten as follows:
4.2 System modeling

To illustrate the effectiveness of the sensitive fault observer, we consider a nonlinear model of the three tank system as follows:

\[
x(k + 1) = \sum_{i=1}^{8} \mu_i(\xi) (A_i x(k)) x(k) + B_i u(k))
\]

(29)

The output of the system is:

\[
y(k) = C x(k) + D_f f(k)
\]

where \(x(k) = (n_1(k) n_2(k) n_3(k))^T\).

The matrices \(A_i\), \(B_i\) and \(C\) are

\[
\begin{bmatrix}
A_{12} & A_{12}^T Z_2 & 0 & C^T V^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
- P_2 & 0 & 0 & Z_2^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta^2 I + 2 G_1(V, V_1^T) V D_s
\end{bmatrix}
\]

\[
\begin{bmatrix}
- I & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
C^T R^{-1} P_2 & C^T R^{-1} P_2 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
R^{-1} P_2 & 0 & 0 & 0
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(28)
\]

Applying Schur complement theorem and assuming that \(Z_2 = P_2 R^{-1}\), we can get the following LMI easily:

\[
\begin{bmatrix}
A_{12} & A_{12}^T Z_2 & 0 & C^T V^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
- P_2 & 0 & 0 & Z_2^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta^2 I + 2 G_1(V, V_1^T) V D_s
\end{bmatrix}
\]

\[
\begin{bmatrix}
- I & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
C^T R^{-1} P_2 & C^T R^{-1} P_2 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
R^{-1} P_2 & 0 & 0 & 0
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(28)
\]

4. THE THREE TANK SYSTEM

In this section, we present the laboratory system that will be used to test the proposed methodology. Its nonlinear model is also derived.

4.1 Process description

Consider the three tanks system shown in Fig 1[6]. The system is composed of three tanks with instrumentation and a deposit. Two pumps permit the water supply of tanks R1 and R2 by adjustment of the control inputs \(u_1\) and \(u_2\), respectively. These tanks R1 and R2 are also connected through a manual valve that makes possible to change the dynamics when desired. Two electronic valves EV1 and EV2 are used to fill tank R3, and an electronic valve EV3 is used to evacuate the water. The water levels are measured with ultrasonic sensors in each tank.

![Fig 1: Three tanks system](image)
The residual weighting matrix \( V \) is
\[
V = \begin{bmatrix}
20.5160 & 0.0542 & -0.1315 \\
0.0541 & 20.4744 & -0.1081 \\
-0.1316 & -0.1082 & 20.6563
\end{bmatrix}
\]

The considered faults sensors are \( f_s = \begin{bmatrix} f_{s1} \\ f_{s2} \\ f_{s3} \end{bmatrix} \)
\[
f_{s1} = \begin{cases} 
0 & k < 60 \\
0.15k & 60 \leq k < 80 \\
0 & k \geq 80
\end{cases}
\]
\[
f_{s2} = \begin{cases} 
0 & k < 60 \\
2\sin(5\times(k-6)) & 60 \leq k < 80 \\
0 & k \geq 80
\end{cases}
\]
\[
f_{s3} = \begin{cases} 
0 & k < 60 \\
0.1(k-2) & 60 \leq k < 70 \\
0 & k \geq 70
\end{cases}
\]

5. SIMULATED AND EXPERIMENTAL RESULTS

Some simulation results are shown in Figures 2-3 via the proposed sensitive fault observer. First, the estimations of the states are plotted in Fig 2.

Fig 3 shows the sensor faults and their estimates.

Some experimental results are shown in Fig 4-5. Fig 4 compares the evolution of the levels in the three tanks with their estimations: it can be seen that the observer provides adequate estimation of the states, even in the presence of faults (presented in Fig 5).

Fig 2: Evolution of the tanks levels and their estimates

Fig 3: Evolution of the faults and their estimation
In summary, by examining the trajectories presented in Fig 2-5, we can confirm that the observer sensitive to the fault gives a good performance. The proposed method supplies adequate estimations of the state of the system and the faults sensors. The obtained results show the efficiency of the proposed approach.

6. CONCLUSION

In this paper, the design of an $H_\infty$ fault detection observer for Takagi-Sugeno discrete-time systems is investigated. This approach allows ensuring the sensitivity via the sensor faults, by using a descriptor systems technique. Sufficient conditions for the existence of a robust observer are expressed in terms of Linear Matrix Inequalities (LMI), by using a quadratic Lyapunov function. Indeed, the design condition ensures the convergence of the observer, guaranteeing an $H_\infty$ performance.

Finally, we have applied the proposed approach on a laboratory plant (a three tanks system). The simulation and experimental results show that the proposed observer gives good estimates for the system states and the sensor faults. Thus, we can confirm the effectiveness of the proposed method.

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8. REFERENCES


