Commuting maps in *G* – metric spaces

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ABSTRACT

In this paper, we prove a common fixed point theorem for commuting maps in *G* – metric spaces.

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INTRODUCTION

In 1992, Dhage[1] introduced the concept of *D* – metric space. Recently, Mustafa and Sims[4] shown that most of the results concerning Dhage's *D* – metric spaces are invalid. Therefore, they introduced a improved version of the generalized metric space structure and called it as *G* – metric space. For more details on *G* – metric spaces, one can refer to the papers [4]-[7].

Now we give basic definitions and some basic results ([4]-[7]) which are helpful for proving our main result.

In 2006, Mustafa and Sims[5] introduced the concept of *G*-metric spaces as follows:

Definition 1.1.[5] Let X be a nonempty set, and let G: X × X × X $\rightarrow$ R+ be a function satisfying the following axioms:

(G1) G(x, y, z) = 0 if x = y = z,

(G2) 0 < G(x, x, y), for all $x, y \in X$ with x ≠ y,

(G3) G(x, x, y) ≤ G(x, y, z), for all $x, y, z \in X$ with z ≠ y,

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = … (symmetry in all three variables) and

(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z) for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G – metric on X and the pair (X, G) is called a G – metric space.

Definition 1.2.[5] Let (X, G) be a G-metric space then for x0 $\in$ X, r > 0, the G-ball with centre x0 and radius r is

BG(x0, r) = {y $\in$ X : G(x0, y, y) < r }.

Proposition 1.1.[5] Let (X, G) be a G-metric space then for any x0 $\in$ X, r > 0, we have,

(1) if G(x0, x, y) < r then x , y $\in$ BG(x0, r),

(2) if y $\in$ BG(x0, r) then there exists a δ > 0 such that BG(y, δ) $\subseteq$ BG(x0, r).

It follows from (2) of the above proposition that the family of all G-balls,

B = {BG(x, r): x $\in$ X, r > 0} is the base of a topology τ (G) on X, the G-metric topology.

Proposition 1.2.[5] Let (X, G) be a G-metric space then for all x0 $\in$ X and r > 0, we have,

$$BG(x0, \frac{1}{3} r) \subseteq B_{d_G}(x0, r) \subseteq BG(x0, r)$$

where dG(x,y) = G(x,y,y) + G(x,x,y), for all $x, y \in X$.

Consequently, the G-metric topology τ (G) coincides with the metric topology arising from dG. Thus, while 'isometrically' distinct, every G-metric space is topologically equivalent to a metric space. This allows us to readily transport many results from metric spaces into G-metric spaces settings.

Definition 1.3.[5] Let (X, G) be a G–metric space, and let {xn} a sequence of points in X, a point 'x' in X is said to be the limit of the sequence {xn} if $\lim_{m,n\to\infty}$ G(x, xn, xm) = 0, and one says that sequence {xn} is G–convergent to x.

Thus, that if xn $\rightarrow$ x or $\lim_{n\to\infty}$ xn = x in a G-metric space (X, G) then for each $\in > 0$, there exists a positive integer N such that G (x, xn, xm) < $\in$ for all m, n ≥ N.

Proposition 1.3.[5] Let *(X, G)* be a *G* – metric space. Then the following are equivalent:

(1) $\{x_n\}$ is *G*-convergent to *x*,

(2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,

(3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$,

(4) $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.

Definition 1.4.[5] Let *(X, G)* be a *G* – metric space. A sequence $\{x_n\}$ is called *G* – Cauchy if, for each $\in > 0$ there exists a positive integer *N* such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$; i.e. if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Proposition 1.4.[5] If *(X, G)* is a *G* – metric space then the following are equivalent:

(1) The sequence $\{x_n\}$ is *G* – Cauchy,

(2) for each $\in > 0$, there exist a positive integer N such that $G(x_n, x_m, x_m) < \in$ for all $n, m \geq N$.

Proposition 1.5.[5] Let *(X, G)* be a *G* – metric space. Then the function *G(x, y, z)* is jointly continuous in all three of its variables.

Definition 1.5.[5] A *G* – metric space *(X, G)* is said to be *G*–complete if every *G*-Cauchy sequence in *(X, G)* is *G*-convergent in *X*.

Proposition 1.6.[5] A G – metric space *(X, G)* is *G* – complete if and only if (X, d_G) is a complete metric space.

Proposition 1.7.[5] Let *(X, G)* be a *G* – metric space. Then, for any $x, y, z, a \in X$, it follows that:

(i) If $G(x, y, z) = 0$, then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq {}^2/_3\,(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

MAIN RESULTS

There has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the "fixed point arena" when the notion of commutativity was used by Jungck [2] to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors. In particular, now we look in the context of common fixed point theorem in G- metric spaces. Start with the following contraction conditions:

Let T be a mapping from a complete G-metric space (X, G) into itself and consider the following conditions:

(1.1) G(Tx, Ty, Tz) ≤ α G(x , y, z) for all $x, y, z \in X$, where $0 \leq \alpha < 1$,

It is clear that every self mapping T of X satisfying condition (1.1) is continuous. Now we focus to generalize the condition (1.1) for a pair of self maps S and T of X in the following way:

(1.2) G(Sx, Sy, Sz) ≤ α G(Tx ,Ty, Tz) for all $x, y, z \in X$, where $0 \leq \alpha < 1$,

To prove the existence of common fixed points for (1.2), it is necessary to add additional assumptions of the following type:

(i) construction of the sequence {xn} (ii) some mechanism to obtain common fixed point and this problem was overcome by imposing additional hypothesis on a pair of { S, T }.

Most of the theorems followed a similar pattern of maps:

(i) contraction (ii) continuity of functions (either one or both) and (iii) some conditions on pair of mappings were given. In some cases, condition (ii) can be relaxed but condition (i) and (iii) are unavoidable.

Definition 2.1. Two mappings *f* and *g* are said to be commuting maps if

$fgx = gfx$ for all $x \in X$.

Now we shall prove our main result:

Theorem 2.1. Let (X, G) be a complete G – metric space and let f and g be self-mappings on X satisfying the following conditions:

(2.1) $f(X) \subseteq g(X)$;

(2.2) g is continuous;

(2.3) $G(fx, fy, fz) \leq q\ G(gx, gy, gz)$, for every $x, y, z \in X$ and $0 < q < 1$.

Then f and g have a unique common fixed point in X provided f and g commute.

Proof. Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such

that $fx_0 = gx_1$, In general choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$.

Now, we prove $\{y_n\}$ is a G-Cauchy sequence in X.

From (2.3), take $x = x_n, y = x_{n+1}, z = x_{n+1}$ we have

$G(fx_n, fx_{n+1}, fx_{n+1}) \leq q\ G(gx_n, gx_{n+1}, gx_{n+1}) = q\ G(fx_{n-1}, fx_n, fx_n)$

Continuing in the same way, we have

$G(fx_n, fx_{n+1}, fx_{n+1}) \leq q^n\ G(fx_0, fx_1, fx_1) \Rightarrow G(y_n, y_{n+1}, y_{n+1}) \leq q^n\ G(y_0, y_1, y_1)$

Therefore, for all $n, m \in N$ (set of natural numbers), $n < m$, we have by $G(5)$

$G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1})+G(y_{n+1}, y_{n+2}, y_{n+2})+G(y_{n+2}, y_{n+3}, y_{n+3})+\text{---}+ G(y_{m-1}, y_m, y_m)$

$$\leq (q^n + q^{n+1} + q^{n+2} + \text{- - -} + q^{m-1})\ G(y_0, y_1, y_1)$$

$$\leq (q^n + q^{n+1} + q^{n+2} + \text{- - -})\ G(y_0, y_1, y_1)$$

$$= \frac{q^n}{(1-q)} G(y_0, y_1, y_1) \to 0 \text{ as } n \to \infty.$$

Thus $\{y_n\}$ is a G – Cauchy sequence in X. Since (X, G) is complete G – metric space, therefore, there exists a point z in X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$. Since g is continuous. Therefore $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = gz$. Further, we have since f and g are commuting maps, therefore by definition, we get $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = gz$. From (2.3), take $x = gx_n$, $y = x_n$, $z = x_n$, we have

$G(fgx_n, fx_n, fx_n) \leq q\ G(ggx_n, gx_n, gx_n)$.

Proceeding limit as $n \to \infty$, we have $z = gz$. We now prove that $z = fz$.

Again from (2.3), setting $x = x_n, y = z, z = z$, we have

$$G(fx_n, fz, fz) \leq q\ G(gx_n, gz, gz) .$$

Taking limit as $n \to \infty$, we have $z = fz$. Therefore, we have $gz = fz = z$. Thus z is a common fixed point of f and g.

Uniqueness. We assume that $z_1\ (\neq z)$ be another common fixed point of f and g.

Then $G(z, z_1, z_1) > 0$ and

$G(z, z_1, z_1) = G(fz, fz_1, fz_1) \leq q\ G(gz, gz_1, gz_1) = q\ G(z, z_1, z_1) < G(z, z_1, z_1)$,

a contradiction, therefore $z = z_1$. Hence uniqueness follows.

Example 2.1. Let $X = [-1, 1]$ and let $G: X \times X \times X \to R^+$ be the G – metric defined as

follows: $G(x, y, z) = \left(|x-y|+|y-z|+|z-x|\right)$ for all $x, y, z \in X$. Then (X, G) is a

G-metric space. Define $f(x) = \frac{x}{6}$ and $g(x) = \frac{x}{2}$. Here we note that,

(1) $f(X) \subseteq g(X)$, (2) g is continuous on X,

(3) $G(fx, fy, fz) \leq q\ G(gx, gu, gz)$, holds for all $x, y, z \in X$, $\frac{1}{3} \leq q < 1$.

However, the maps *f* and *g* are commuting maps and $x = 0$ is the unique common fixed point of *f* and *g*. Thus all the conditions of the Theorem 2.1 are satisfied.

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