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A Numerical Algorithm for Solution of Boundary Value Problems with Applications

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ABSTRACT
A numerical method is presented in this paper which employs cubic B-spline to solve two point second order boundary value problems for ordinary differential equations. First, heat problem is modeled as second order boundary value problem. Then, B-spline method for both linear and non-linear cases is discussed. Selected numerical examples for both the cases are solved using *MATLAB*, which demonstrate the applicability and efficiency of present method. To be more accessible for practicing engineers and applied mathematicians there is a need for methods, which are easy and ready for computer implementation. The B-spline techniques appear to be an ideal tool to attain these goals. An added advantage of present method is that it does not require modification while switching from linear to non-linear problem.

General Terms
Computational Mathematics

Keywords
Boundary value problem (BVP), linear and non-linear differential equation, Cubic B- spline, nodal points, heat flow.

1. INTRODUCTION
Second order two point Boundary value problems are encountered in many engineering fields including optimal control, beam deflections, heat flow, and various dynamical systems. The mathematical modeling of heat flow problem is presented here as an illustration [1].

1.1 Heat problem:

Consider the temperature distribution within a rod of uniform cross section that conducts heat from one end to the other. (Fig. 1)

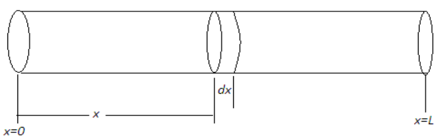

Fig. 1: Temperature distribution in a rod

By concentrating our attention on an element of the rod of length *dx* located at a distance *x* from the left end, we can derive the equation that determines the temperature, *u*, at any point along the rod. The rod is perfectly insulated around its outer circumference so that heat flows only laterally along the rod. It is well known that heat flows at a rate proportional to the cross section area (*A*), to a property of the material [*k*, its thermal conductivity], and to the temperature gradient, *du/dx*, at point *x*. We use *u(x)* for the temperature at point *x* measured from the left end of the rod. Thus the rate of flow of heat into the element is $-kA\left(\frac{du}{dx}\right)$. The rate at which heat leaves the element is given by a similar equation, but now the temperature gradient must be at the point $x+dx: -kA\left[\frac{du}{dx}+\frac{d}{dx}\left(\frac{du}{dx}\right)dx\right]$, unless heat is being added to the element (or withdrawn by some means), the rate that heat flows from the element must equal the rate that heat enters, or else the temperature of the element will vary with time. In case of steady-state or equilibrium temperatures, we can equate the rates of heat entering and leaving the element: $-kA\left(\frac{du}{dx}\right)=-kA\left[\frac{du}{dx}+\frac{d}{dx}\left(\frac{du}{dx}\right)dx\right]$. When some common terms on each side of the equation are canceled, we get very simple relation $kA\frac{d}{dx}\left(\frac{du}{dx}\right)dx=kA\frac{d^2u}{dx^2}=0$, where we have written the second derivative in its usual form. For this particularly simple example, the equation for *u* as a function of *x* is the solution to $\frac{d^2u}{dx^2}=0$ and this is obviously just $u=ax+b$, a linear relation.

The rod could also lose heat from the outer surface of the element. If this is *Q*, the rate of heat flow in must equal the rate leaving the element by conduction along the rod plus the rate at which heat is lost from the surface. This means that

$$-kA\left(\frac{du}{dx}\right)=-kA\left[\frac{du}{dx}+\frac{d}{dx}\left(\frac{du}{dx}\right)dx\right]+Qpdx \tag{1}$$

Where *p* is the perimeter at point *x*. If this equation is expanded and common terms are canceled, we get a somewhat more complicated equation whose solution is not obvious: $\frac{d^2u}{dx^2}=\frac{Qp}{kA}$, where *Q* can be a function of *x*. Now, suppose that the cross section varies with *x*. We will have, then, for the rate of heat leaving the element

$$-k[A+A'dx]\left[\frac{du}{dx}+u''dx\right] \tag{2}$$

Where we have used a prime notation for derivatives with respect to *x*. Equating the rates in and out as before and cancelling common terms results in

$$kAu''dx+kA'u'dx+kA'u''dx^2=Qpdx \tag{3}$$

We can simplify this further by dropping the term with dx^2 because it goes to zero faster than the terms in dx . After also dividing out *dx*, this results in a second order differential equation of the form

$$kAu''+kA'u'=Qp \tag{4}$$

The equation can be generalized even more if *k* also varies along the rod. In this case the ODE becomes

$$kAu''+(kA'+k'A)u'=Qp \tag{5}$$

If the rate of heat loss from the outer surface is proportional to the difference in temperatures between that within the element and the surroundings (u_s), we must substitute $Q=q(u-u_s)$ giving

$$kAu''+(kA'+k'A)u'-q*pu=-q*pu_s \tag{6}$$

We can rewrite this equation as

$$a_1(x)\frac{d^2u}{dx^2}+a_2(x)\frac{du}{dx}+a_3(x)u=f(x) \tag{7}$$

Where $a_1(x), a_2(x), a_3(x)$ and $f(x)$ are functions of *x* only. (***Linear*** 2nd order ODE)

In temperature distribution, *non-linearity* can be caused if the thermal conductivity, *k*, is considered to vary with the temperature, *u*. To solve equation (7), we must know two conditions on *u* or its derivative. If both *u* and u' are specified at some starting value for *x*, the problem is an initial value problem. However, if two values of *u* and/or u' are given at two different values of *x*, the problem becomes a two point boundary value problem.

Theorem (Existence of solution): For the boundary value problem written in the form $u''=p(x)u'+q(x)u+r(x)$, $a<x<b$ with $u(a)=\alpha$, $u(b)=\beta$, a sufficient condition for the existence and uniqueness of the solution of problem is: *If* $p(x)$ *,* $q(x)$ *and* $r(x)$ *are real valued functions defined on [a, b] with q(x)>0 for* $x\in[a,b]$*, then the boundary value problem has a unique solution for each* α *and* β . [2]

Various numerical methods have been developed for solution of boundary value problems (BVPs). Two well known methods are finite difference methods and spline methods. Splines (polynomial as well as non-polynomial) of various degrees have been used by many authors to solve BVPs (see [3, 4, 5] and references therein). Most of the above mentioned methods concentrate on the special case linear second order BVP only. The present paper, however, delivers a numerical method based on cubic B-spline for general linear second order two point BVP as well as non-linear second order BVP.

Present paper is organized as follows: Section 2 contains cubic B-spline method for linear general case second order BVP. In Section 3, B-spline method is applied for non-linear case. Numerical examples are given in Section 4 followed by conclusion in Section 5.

2. CUBIC B-SPLINE METHOD FOR SECOND ORDER LINEAR BOUNDARY VALUE PROBLEM

Consider linear boundary value problem (8) in the form

$$a_1(x)u''+a_2(x)u'+a_3(x)u=f(x) \tag{8}$$

Subject to boundary conditions

$$u(a)=u_0,\quad u(b)=u_L \tag{9}$$

We subdivide the interval [a, b] and choose piecewise uniform grid points represented by $\prod : x_0<x_1<....<x_n$, such that $x_0=a, x_n=b$ and h is the piecewise uniform spacing. Let $S_3(\prod)$ be the space of cubic spline functions over the partition $\prod$. We can define the cubic B-spline basis functions $\{B_i(x)\}$, for $i=-1,0,1,......,n+1$, for $S_3(\prod)$ after including two more points on each side of the partition $\prod$. Thus the partition $\prod$ becomes

$$\prod : x_{-2}<x_{-1}<x_0<x_1<....<x_n<x_{n+1}<x_{n+2}$$

Now, the cubic B-Spline basis function is defined as,

$$B_i(x)=\frac{1}{h^3}\begin{cases}(x-x_{i-2})^3, & if\ x\in[x_{i-2},x_{i-1}]\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2 & \\ -3(x-x_{i-1})^3, & if\ x\in[x_{i-1},x_i]\\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2 & \\ -3(x_{i+1}-x)^3, & if\ x\in[x_i,x_{i+1}]\\ (x_{i+2}-x)^3, & if\ x\in[x_{i+1},x_{i+2}]\\ 0, & otherwise.\end{cases} \tag{10}$$

It can easily be verified from (10) that each of the functions $B_i(x)$ is twice continuously differentiable on the entire real line.

Also, we have the followings from the above definition,

$$B_i(x_k)=\begin{cases}4, & if\ i=k\\ 1, & if\ i-k=\pm1\\ 0, & if\ i-k=\pm2\end{cases} \tag{11}$$

$$B_i'(x_k)=\begin{cases}0, & if\ i=k\\ \pm\frac{3}{h}, & if\ i-k=\pm1\\ 0, & if\ i-k=\pm2\end{cases} \tag{12}$$

$$B_i''(x_k)=\begin{cases}\frac{-12}{h^2}, & if\ i=k\\ \frac{6}{h^2}, & if\ i-k=\pm1\\ 0, & if\ i-k=\pm2\end{cases} \tag{13}$$

Now, suppose $S_3(\prod) = span\{B_{-1}(x), B_0(x), ..., B_n(x), B_{n+1}(x)\}$. It has been proved that [6], all $B_i(x)'s$ are linearly independent and $\dim S_3(\prod) = n+3$. Further, let *S(x)* be the B-spline interpolating the function *u(x)* at the nodal points and $S(x) \in S_3(\prod)$, then we have

$$u(x) = S(x) = \sum_{j=-1}^{n+1} c_j B_j(x) \tag{14}$$

Let the approximate solution of given BVP is given by (14), where $c_j's$ are unknown coefficients and $B_j(x)$'s third degree B-Spline functions.

Now, this approximate solution must satisfy the given BVP at the nodal points $x = x_i$. For, putting values from (14) in (8), we get

$$\sum_{j=-1}^{n+1} c_j a_1(x_i) B''_j(x_i) + \sum_{j=-1}^{n+1} c_j a_2(x_i) B'_j(x_i) + \sum_{j=-1}^{n+1} c_j a_3(x_i) B_j(x_i) = f(x_i), \quad i = 0,1,2,....,n \tag{15}$$

And boundary conditions give

$$\sum_{j=-1,0,1} c_j B_j(x_0) = u_0 \tag{16}$$

And $$\sum_{j=n-1,n,n+1} c_j B_j(x_n) = u_L \tag{17}$$

The values of spline functions and derivatives at the knots are determined using relations (11-13) and substituting these in (15)-(17). Then a system of $(n+3)$ linear equations in $(n+3)$ unknowns $c_{-1}, c_0, ..., c_{n+1}$ is obtained. This system can be written in matrix vector form as

$$AC = F \tag{18}$$

Where

$$C = \left[c_{-1}, c_0, ..., c_{n+1}\right]^T \tag{19}$$

$$F = \left[u_0, f(x_0), ..., f(x_n), u_L\right]^T \tag{20}$$

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & ... & 0 & 0 \\ \alpha(x_0) & \beta(x_0) & \gamma(x_0) & 0 & ... & 0 & 0 \\ 0 & \alpha(x_1) & \beta(x_1) & \gamma(x_1) & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & ... & 0 & 0 & \alpha(x_n) & \beta(x_n) & \gamma(x_n) \\ . & . & . & . & 1 & 4 & 1 \end{bmatrix} \tag{21}$$

and elements of A are given by,

$$\alpha(x_i) = a_1(x_i)\frac{6}{h^2} + a_2(x_i)\frac{3}{h} + a_3(x_i), \qquad i = 0,1,........,n \tag{22}$$

$$\beta(x_i) = a_1(x_i)\frac{-12}{h^2} + a_2(x_i)\frac{0}{h} + 4a_3(x_i), \quad i = 0,1,........,n \tag{23}$$

$$\gamma(x_i) = a_1(x_i)\frac{6}{h^2} + a_2(x_i)\frac{-3}{h} + a_3(x_i), \qquad i = 0,1,........,n \tag{24}$$

The B-spline solution is obtained by solving the above system.

3. CUBIC B-SPLINE METHOD FOR NON- LINEAR BOUNDARY VALUE PROBLEM

While the general theory and methods for linear differential equations are highly developed, little of a general character is known about nonlinear equations. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation. Cubic B-spline method for second order non-linear boundary value problem is outlined here.
Consider the second order boundary value problem given by

$$u'' + f(x)u' = g(x,u) \tag{25}$$

Where $g(x,u)$ is non-linear and boundary conditions are given by (9).

Proceeding as in previous section, Let $u(x) = \sum_{j=-1}^{n+1} c_j B_j(x)$ be the approximate solution of given BVP, which must satisfy the given system at the nodal points $x = x_i$. For, putting values in (25), we get a non-linear system of equations given by

$$\sum_{j=-1}^{n+1} c_j B''_j(x_i) + \sum_{j=-1}^{n+1} c_j f(x_i) B'_j(x_i) = g\left(x_i, \sum_{j=-1}^{n+1} c_j B_j(x_i)\right) \tag{26}$$

$$\sum_{j=-1}^{n+1} c_j B_j(x) = u_0, \;\; for\; x = a \tag{27}$$

$$\sum_{j=-1}^{n+1} c_j B_j(x) = u_L, \quad for\; x = b \tag{28}$$

The above non-linear system can be solved by Newton method to get the solution.

4. NUMERICAL EXAMPLES

Problem 1. Consider

$$\frac{d^2u}{dx^2} = 100u \quad \text{with } u(0) = u(1) = 1 \tag{29}$$

The exact solution is $u(x) = \cosh(10x-5)/\cosh 5$.

The numerical results are compared with that of [7] in Table 1.

Table 1: Maximum Absolute Errors for Problem 1

n	Maximum absolute error by al said [7]	Maximum absolute error by present method
10	*N.A.*	$2.04E-6$
20	$1.57E-3$	$4.38E-7$
40	$4.53E-4$	$8.99E-8$

Problem 2. Consider $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u = -3$ (30)

with $u(0) = -3,\ u(1) = -2.26424$

The exact solution is $u(x) = 2xe^{x-2} - 3$.

The numerical results are given in Table 2.

Table 2: Maximum Absolute Errors for Problem 2

n	Maximum absolute error by present method
10	$8.37E-7$
20	$2.17E-7$
40	$5.09E-8$

Problem 3. Consider

$$u'' + u^2 - x^4 = 2 \quad \text{with } u(0) = 0,\ u(1) = 1 \tag{31}$$

The exact solution is $u(x) = x^2$.

The numerical results are given in Table 3.

Table 3: Maximum Absolute Errors for Problem 3

n	Maximum absolute error by HPM method [8]	Maximum absolute error by present method
10	$1.02E-3$	$1.32E-6$
20	*N.A.*	$2.58E-7$
40	*N.A.*	$5.31E-8$

Problem 4. (Special case of Troesch's problem)

$$\frac{d^2u(x)}{dx^2} = \sinh u(x) \quad \text{with} \quad u(0) = 0,\ u(1) = 1 \tag{32}$$

The numerical results are given in Table 4.

Table 4: Maximum Absolute Errors for Problem 4

n	Maximum absolute error by HPM method [8]	Maximum absolute error by present method
10	$1.78E-2$	$2.11E-5$
20	*N.A.*	$4.39E-6$
40	*N.A.*	$1.11E-7$

5. CONCLUSION

We conclude by taking note that the present study furnished a numerical treatment for linear and non-linear two-point boundary value problems by B-spline method, which approximate the exact solutions well. In addition, the present B-spline method is of versatile nature which does not require modification while switching from linear to non-linear problem.

6. REFERENCES

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