Characterizations of T - Fuzzy R – Ideals on BCI-Algebras

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ABSTRACT
The fuzzy R-ideals of BCI-algebras are studied, and related properties are investigated.

Key words: BCI –algebra's, R-ideals, imaginalble, T-fuzzy R-ideals

1. INTRODUCTION
BCK/BCI-algebras are two important classes of logical algebras introduced by Iseki in 1966 (see [5, 6, 14]) since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras (see [15]). Meng [15] introduced the notions of implicative ideals and commutative ideals in BCK-algebras and applied them to characterize implicative BCK-algebras and commutative BCK-algebras, respectively. Meng also showed that a non-empty subset of a BCK-algebra is an implicative ideal if and only if it is both a commutative ideal and a positive implicative ideal. All the above results mo-tivate us to further investigate the relations between algebras and ideals and between ideals and ideals. The notion of quasi-associative BCI-algebras was introduced by Xi [19]. They are all important classes of BCI-algebras. In [21], the notion of p-ideals was introduced and used to characterized-semisimple BCI- algebras.

The concept of fuzzy subset and various operations on it were first introduced by Zadeh in [20]. Since then, fuzzy subsets have been applied to diverse field. The study of fuzzy subsets and their application to mathematical contexts has reached to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy sub-groups in 1971 by Rosenfeld [17]. Since then these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. A. KORDI AND A. MOUSSAVI 1999, Ougen [16] defined fuzzy subsets in BCK-algebras and investigated some properties. In 1993, Jun [8] applied it in BCI-algebras. Fuzzy p-ideals and fuzzy H-ideals in BCI-algebras is introduced in [9], [10] respectively and several interesting properties of these concepts are studied. Following [9] and [10], we study and give some characterizations of these ideals. BCK/BCI-algebras are two important classes of logical algebras introduced by Iseki in 1966 [5, 6].

2. PRELIMINARIES
Definition2.1: An algebra $(X; *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Definition2.2: In a BCI-algebra X, a partially ordered relation $\leq$ can be defined by

$x \leq y$ if and only if $x * y = 0$.

Definition2.3: A BCI-Algebra X satisfies the following properties:

(1.1) $(x * y) * z = (x * z) * y$,

(1.2) $x * 0 = x$,

(1.3) $0 * (x * y) = (0 * x) * (0 * y)$,

(1.4) $0 * (0 * (x * y)) = 0 * (y * x)$,

(1.5) $(x * z) * (y * z) \leq x * y$,

(1.6) $x * y = 0$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Definition2.4: A subset A of a BCI-algebra X is called an ideal if (1) $0 \in A$,(2) for any $x, y \in X$, $x * y, y \in A$ imply $x \in A$.

Definition2.5: (Jun [8]) an ideal A in a BCI-algebra X is said to be closed if for all $x \in X$, $0 * x \in A$ implies $x \in A$.

Definition2.6: (Zadeh [20]) Let X is a set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$.

Notation2.7: Let μ and γ be fuzzy sets in a BCI-algebra X. By $\mu \leq \gamma$ we mean that $\mu(x) \leq \gamma(x)$ for any $x \in X$.

Definition2.8: ([8]) Let μ be a fuzzy subset of a BCI-algebra X. For $t \in [0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ.

Definition2.9: [8] A fuzzy set μ in a BCI-algebra X is said to be a fuzzy ideal in X if it satisfies (F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Definition2.10: (Jun [8]) A fuzzy ideal μ in a BCI-algebra X is said to be closed if for all $x \in X$, $\mu(0 * x) \geq \mu(x)$.

Proposition2.11:[13] Let μ be a fuzzy ideal in a BCI-algebra X. Then $x \leq y$ implies $\mu(y) \leq \mu(x)$.

Proposition2.12: ([13]) A fuzzy set μ satisfying (F1) in a BCI-algebra X is a fuzzy ideal if and only if for all x, y, z ∈ X, $(x * y) * z = 0$ implies $\mu(x) \geq \min\{\mu(y),\mu(z)\}$.

3. PROPERTIES OF T- FUZZY R-IDEALS

Theorem3.1: Every imaginable T –fuzzy R –ideal of a BCI – algebras X is a fuzzy R –ideal

Proof: Let A be an imaginable T –fuzzy R- ideal with membership function

μ_A. Then for all x, y, z ∈ X. $\mu_A(x) = T\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$.

Since A is imaginable, it follows that

$\min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$

$= T\{\min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}, \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}\}$

$\leq T\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$

$= \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$.

It implies that $\mu(x) \geq T\{\mu_A((x*z)*(z*y)), \mu_A(y)\} = \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$. On the other hand since μ_A is T–fuzzy R–ideal. Thus $\mu_A(0) = \mu_A(x)$. so that A is a fuzzy R – ideal of X .

Theorem3.2: Every T –fuzzy R –ideal in a BCI – algebras X is a T-fuzzy ideal of X.

Proof: For all x, y, z, we have $\mu_A(x) \geq T\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$

Putting z = y $\mu_A(x) = T\{\mu_A((x*y)*(y*y)), \mu_A(y)\} = T\{\mu_A((x*y)*(0)), \mu_A(y)\}$

$\mu_A(x) \geq T\{\mu_A(x*y), \mu_A(y)\}$ which completes the proof.

Theorem3.3: Every fuzzy R–ideal in a BCI – algebras X is a fuzzy ideal of X.

Proof: For all x, y, z, we have $\mu_A(x) \geq \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$

Putting z = y $\mu_A(x) = \min\{\mu_A((x*y)*(y*y)), \mu_A(y)\} = \min\{\mu_A((x*y)*(0)), \mu_A(y)\}$

$\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$ which complete the proof.

Theorem3.4: If μ is a T –fuzzy R –ideal in a BCI – algebras X, then each non-empty level subset is R- ideal of X.

Proof: suppose that μ is a T- fuzzy R- ideal of X since $\cup(\mu: 1)$ is non-empty, there exists $x \in \cup(\mu: 1)$. It follows forms that $\mu(0) \geq \mu(x) = 1$ $(0 \in \cup(\mu: \alpha)$.

Let x, y, z ∈X be such that $(x*z)*(z*y)$ x $\cup(\mu: 1)$ and $y \in \cup(\mu: 1)$.

Then $\mu_A(x) = T\{\mu_A((x*y)*(y*y)), \mu_A(y)\} = T(1, 1) = 1$.so that $y * x \in \cup(\mu: 1)$.

Hence $\cup(\mu: 1)$ is a R- ideal of X.

Theorem3.5: If λ and μ are T- fuzzy R- ideal of a BCI-algebra X, then $\lambda \times \mu$ is a T –fuzzy R- ideal of X.

Proof: For any $(x, y) \in X \times Y$. we have $(\lambda \times \mu)(0, 0) = T\{\lambda(0), \mu(0)\} \geq T\{\lambda(x), \mu(y)\}$.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2) \in X \times Y$.

Then $(\lambda \times \mu)(x) = (\lambda \times \mu)(x_1, x_2) = T\{\lambda(x_1), \lambda(x_2)\}$

$\geq T[T\{\lambda((x_1*z_1)*(z_1*y_1)), \lambda(y_1)\}, T\{\mu\{((x_1*z_2)*(z_2*y_2)), \mu(y_2)\}]$

$=T[T\{\lambda((x_1*z_1)*(z_1*y_1)), ((x_1*z_2)*(z_2*y_2))\}, T\{\lambda(y_1), \mu(y_2)\}]$

$=T[(\lambda \times \mu)\{((x_1*z_1)*(z_1*y_1)), ((x_1*z_2)*(z_2*y_2))\}, (\lambda \times \mu)\{y_1, y_2\}]$

$=T[(\lambda \times \mu)\{((x_1, x_2)*(z_1, z_2))*((z_1, z_2)*(y_1, y_2))\}, (\lambda \times \mu)\{y_1, y_2\}]$

$=T[(\lambda \times \mu)((x*z)*(z*y)), (\lambda \times \mu)(y)]$

Hence $\lambda \times \mu$ is a T- fuzzy R- ideal of $x \times y$

Theorem3.6: If A is a fuzzy R-ideal of BCI algebra X, then A^m is a fuzzy R-ideal of X.

Proof: $\mu_A(0) \geq \mu_A(x)$ for all x ε X implies $(\mu_A(0))^m \geq (\mu_A(x))^m$

$\mu_A(0)^m \geq \mu_A(x)^m$, $\mu_A{}^m(0) \geq \mu_A{}^m(x)$ for all x ε X

For the second condition $\mu_A(x) \geq \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}$

$(\mu_A(x))^m \geq [\min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}]^m \mu_A(x)^m \geq \min\{\mu_A((x*z)*(z*y)), \mu_A(y)\}^m$

$\mu_A{}^m(x) \geq \min\{\mu_A((x*z)*(z*y))^m, \mu_A(y)^m\}$

$\mu_A{}^m(x) \geq \min\{\mu_A{}^m((x*z)*(z*y)), \mu_A{}^m(y)\}$ for all x, y εX

Therefore A^m is a fuzzy ideal of X.

Theorem3.7: If μ is a fuzzy R -ideal of BCI algebra X, then μ^m is a fuzzy H- ideal of a BCI-algebra X.

Proof: Now $\mu(0) \geq \mu(x)$

$\mu(0)^m \geq (\mu(x))^m$

$\mu(0)^m \geq \mu(x)^m$

$\mu^M(0) \geq \mu^m(x)$ for all x 0 X

$\mu(x * z) \geq$ Min $[\mu[x*(y*z)], \mu_A(y)]$ for all x, y, z 0 X

$\{\mu(x*z)\}^m \geq \{\min[\mu[x*(y*z)], \mu(y)]\}^m$

$\mu(x*z)^m \geq \min[\mu[x*(y*z)], \mu(y)^m]$

$\mu^{M(}x*z) \geq \min[\mu[x*(y*z)]^m, \mu(y)^m]$

$\mu^{M(}x*z) \geq \min[\mu^m[x*(y*z)], \mu^m(y)]$

μ^M is a fuzzy R- ideal of a BCI-algebra X.

Theorem3.8: If μ_1 & μ_2 are fuzzy R- ideal of BCI – algebra X, if one is contained other then show that $\mu_1 \cup \mu_2$ is Fuzzy R- ideal of X

Proof: i) Now $\mu_1(0) \geq \mu_1(x)$,

$\mu_2(0) \geq \mu_2(x)$, for all x ε X

Max $\{\mu_1(0), \mu_2(0)\} \geq \max\{\mu_1(x), \mu_2(x)\}$

$\mu_1 \cup \mu_2 \geq \min\{\mu_1(x), \mu_2(x)\}$

$\mu_1 \cup \mu_2$ (0) $\geq \mu_1 \cup \mu_2$ (x) for all x 0 X

μ_1(x*z)$\geq$ min [μ_1[x*(y*z)], μ_1 (y)]

μ_2(x*z)$\geq$ min [μ_2[x*(y*z)], μ_2 (y)]

Max {μ_1 (x), μ_2 (x)} $\geq$ max [min [μ_1[x*(y*z)], μ_1 (y)], min [μ_2[x*(y*z)], μ_2 (y)]]

Given that $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

Max {μ_1 (x*z), μ_2 (x*z)} $\geq$ min [max [μ_1[x*(y*z)], μ_2[x*(y*z)], max [μ_1 (y), μ_2 (y)]]

$\mu_1 \cup \mu_2$ (x*z)} $\geq$ min [$\mu_1 \cup \mu_2$[x*(y*z)], [$\mu_1 \cup \mu_2$ (y)]]

Therefore $\mu_1 \cup \mu_2$ is a fuzzy R-ideal of X.

Theorem3.9 : If μ_1 & μ_2 are fuzzy R- ideal of BCI–algebra X, $\mu_1 \cap \mu_2$ is Fuzzy R-ideal of X.

Proof: i) Now μ_1 (0) $\geq \mu_1$(x),

μ_2 (0) $\geq \mu_2$(x), for all x ε X

Min {μ_1 (0), μ_2 (0)} $\geq$ min {μ_1(x), μ_2(x)}

$\mu_1 1 \mu_2 \geq$ min {μ_1(x), μ_2(x)}

$\mu_1 1 \mu_2$ (0) $\geq \mu_1 1 \mu_2$ (x) for all x 0 X

μ_1(x*z)$\geq$ min [μ_1[x*(y*z)], μ_1 (y)]

μ_2(x*z)$\geq$ min [μ_2[x*(y*z)], μ_2 (y)]

Min {μ_1 (x*z), μ_2 (x*z)} $\geq$ min [min [μ_1[x*(y*z)], μ_1 (y)], min [μ_2[x*(y*z)], μ_2 (y)]]

Min {μ_1 (x*z), μ_2 (x*z)} $\geq$ min [min [μ_1[x*(y*z)], μ_2[x*(y*z)], min [μ_1 (y), μ_2 (y)]]

$\mu_1 \cap \mu_2$ (x*z)} $\geq$ min [$\mu_1 \cap \mu_2$[x*(y*z)], [$\mu_1 \cap \mu_2$ (y)]]

Therefore $\mu_1 \cap \mu_2$ is a fuzzy R-ideal of X.

4. CONCLUSION

Y.B.Jun and J.Meng introducedthe concept of P-ideas in BCI algebras .In this paper, we investigate the characterizations of T-fuzzy R-ideals on BCI algebras.

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