Quasi-Cut of Fuzzy Sets and Quasi-Cut of Intuitionistic Fuzzy Sets

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ABSTRACT

The aim of this paper is to study the properties of t-cut set, strong t-cut set, t-quasi-cut set, strong t-quasi-cut set and ∈∨-cut set of an intuitionistic fuzzy set μ. For any intuitionistic fuzzy set A = (μA, νA) > and α, β ∈ [0, 1], we define and study the properties of upper (α, β) cut set A(α, β), strong upper (α, β) cut set A′(α, β), lower (α, β) cut set A(α, β), strong lower (α, β) cut set A′(α, β), upper (α, β)-quasi-cut set A(α, β), lower (α, β)-quasi-cut set A(α, β), strong upper (α, β)-quasi-cut set A′(α, β), strong lower (α, β)-quasi-cut set A′(α, β), and ∈∨-cut set.

Keywords

Cut set, Strong cut set, Quasi cut set, Strong quasi cut set, (α, β)-cut set, Upper (α, β)-cut set, Strong upper (α, β)-cut set, Lower (α, β)-cut set, Strong lower (α, β)-cut set, Upper (α, β)-quasi-cut set, Strong upper (α, β)-quasi-cut set, Strong lower (α, β)-quasi-cut set, ∈∨-cut set.

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1. INTRODUCTION

In many complicated situations in real life several types of uncertainties occur, to handle such situations we have theory of Probability, theory of Interval Mathematics, Fuzzy set theory, Rough set theory and Vague set theory. Fuzzy set is introduced in [1] by Zadeh. The theory of fuzzy set is further generalised to intuitionistic fuzzy set, interval valued fuzzy set, temporal intuitionistic fuzzy set etc. The concept of intuitionistic fuzzy set introduced by Atanassov [2] in 1983. Das [3] introduced the notion of level subset (called cut set) of a set. Since then different researchers [4][5][6][7][8] have contributed significantly for the development of literatures of cut sets.

In the theory of fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets, cut sets play very important rule for the development of the theory. Here in this paper, various types of cut sets in fuzzy sets and in intuitionistic fuzzy sets are discussed.

2. PRELIMINARIES

DEFINITION 1. (4,5) A fuzzy set μ of the form

\[ μ(y) = \begin{cases} t, & \text{if } y = x, t \in (0,1] \\ 0, & \text{if } y \neq x \end{cases} \]

is called a fuzzy point with support x and value t and is denoted by x_t.

DEFINITION 2. (4,5) A fuzzy point x_t is said to belong to (respectively be quasi coincident with) a fuzzy set μ written as x_t ∈ μ (respectively x_t ∈ μ) if μ(x) ≥ t (respectively μ(x) + t > 1). If x_t ∈ μ or x_t ∈ μ, then we write x_t ∈ μ or x_t ∈ μ. (Note ∈ μ means ∈ μ does not hold).

DEFINITION 3. Let X be a set and μ be a fuzzy subset of X, then t-cut set and t-strong cut set of fuzzy set μ are given by

\[ μ_t = \{ x | x \in X \text{ and } μ(x) ≥ t \} \]

From the point of view of neighbourhood, we have x ∈ μ_t ⇔ μ(x) ≥ t ⇔ x_t ∈ μ. Prof.L. Cheng-Zhong [9]introduced a new concept of strong neighbourhood and he define \[ x \in μ_t \Leftrightarrow μ(x) > t \Leftrightarrow x_t \in μ \] Therefore t-cut set and t-strong cut set of fuzzy set μ are given by

\[ μ_t = \{ x | x \in μ \} \quad μ_t = \{ x | x_t \in μ \} \]

Quasi neighbourhood play an important rule in fuzzy topology [11][22]. Now x_t ∈ μ ⇔ μ(x) + t > 1 based on quasi neighbourhood, we can defined a new kind of cut set as \[ μ > t = \{ x | x_t \in μ \} \] Here \( μ > t \) is called a t-strong quasi cut set of fuzzy set μ. Combining both the notion we define another cut set as \[ μ_t = \{ x | x_t \in μ \} \]

DEFINITION 4. Let X be a set and μ be a fuzzy subset of X, then t-quasi-cut set and t-strong quasi cut set of fuzzy set μ are given by

\[ μ < t = \{ x | x \in X \text{ and } μ(x) + t \geq 1 \} = \{ x | x_t \in μ \} \]

\[ μ > t = \{ x | x \in X \text{ and } μ(x) > t \} = \{ x | x_t \in μ \} \]

DEFINITION 5. (9,79) An intuitionistic fuzzy set (IFS) A of a BG-algebra X is an object of the form A = \{ < x, μA(x), νA(x) > | x ∈ X \}, where μA : X → [0, 1] and νA : X → [0, 1] with the condition 0 ≤ μA(x) + νA(x) ≤ 1, ∀x ∈ X. The numbers μA(x) and νA(x) denote respectively the degree of membership and the degree of non-membership of the element x in set A. For the sake of simplicity, we shall use the symbol A = (μA, νA) for the intuitionistic fuzzy set A = {<
under the map $f$ is denoted by fuzzy subsets of $X$ and $Y$ respectively. Then the image of $f(A)$ and is defined by $f = \mu$ where $x \in \mathcal{A}$ is
\[
\nu(x) = \mu(x) | x \in X \}\text{ is denoted by }\nu(x) = \mu(x) | x \in X \}
\]
where $\mu(x)$ and $\nu(x)$ are given by
\[
\mu(x) = \min\{\mu(x), \mu_B(x)\} \quad \text{and} \quad \nu(x) = \mu(x) \quad \text{if } x \in X \}
\]

**Definition 6.** If $A = \{x, \mu_A(x), \nu_A(x) | x \in X \}$ and $B = \{x, \mu_B(x), \nu_B(x) | x \in X \}$ are any two IFs of a set $X$, then $A \subseteq B$ if and only if for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, $A = B$ if and only if for all $x \in X, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, $A \cap B = \{x, (\mu_A \cap \mu_B)(x), (\nu_A \cap \nu_B)(x) | x \in X \}$, where $\mu_A \cap \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$, $A \cup B = \{x, (\mu_A \cup \mu_B)(x), (\nu_A \cup \nu_B)(x) | x \in X \}$, where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cup \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

**Definition 7.** Let $X$ and $Y$ be two non empty sets and $f : X \rightarrow Y$ be a mapping. Let $\mu$ and $\nu$ be two fuzzy subsets of $X$ and $Y$ respectively. Then the image of $\mu$ under the map $f$ is denoted by $f(\mu)$ and is defined by $f(\mu)(y) = \{\forall x \in f^{-1}(y) | \mu(x) \neq 0 \}$ and $\{0 \text{ otherwise} \}$, also a pre image of $\nu$ under $f$ is denoted by $f^{-1}(\nu)$ and is defined as $f^{-1}(\nu)(x) = f(f(x))) | \forall x \in X$

**Definition 8.** Let $X$ and $Y$ be two non empty sets and $f : X \rightarrow Y$ be a mapping. Let $A$ and $B$ be IFs’ of $X$ and $Y$ respectively. Then the image of $A$ under the map $f$ is denoted by $f(A)$ and is defined by $f(A)(y) = \{\forall x \in f^{-1}(y) | \mu(x) \neq 0 \}$ and $\{0 \text{ otherwise} \}$, also a pre image of $B$ under $f$ is denoted by $f^{-1}(B)$ and is defined as $f^{-1}(B)(x) = (\mu(f^{-1}(B)), (\nu(f^{-1}(B))) = (\mu_B(f(x)), \nu_B(f(x))) | \forall x \in X$

\[
\mu_A(x) \leq \mu(f_A)(f(x)) \quad \text{and} \quad \nu_A(x) \geq \nu(f_A)(f(x)) \quad \forall x \in X \}
\]

**Definition 9.** A fuzzy point $x_t$ is said to belong to (respectively be quasi coincident with) an intuitionistic fuzzy set $A = \{x, \mu_A(x), \nu_A(x) | x \in X \}$ written as $x_t \in A$ (respectively $x_t \in A$), if $\mu_A(x) \geq t$ (respectively $\mu_A(x) \geq t$) and $\nu_A(x) \leq t$ (respectively $\nu_A(x) \leq t$). If $x_t \in A$ or $x_t \notin A$, then $x_t \notin A$.

**Definition 10.** Let $A = \{x, \mu_A(x), \nu_A(x) | x \in X \}$ and $t \in [0, 1]$, then t-quasi cut set, t-strong quasi cut set and $\mu$ cut set of fuzzy membership set $\mu_A$ are given by
\[
\mu_A^{-\alpha} = \{x \mu_A(x) \geq \alpha \}
\]
\[
\mu_A^{-\nu} = \{x \mu_A(x) \leq \nu \}
\]
\[
\mu_A^{-\nu} = \{x \mu_A(x) \leq \nu \}
\]
\[
\nu_A^{-\alpha} = \{x \nu_A(x) \geq \alpha \}
\]
\[
\nu_A^{-\nu} = \{x \nu_A(x) \leq \nu \}
\]

\[
\mu_A^{-\mu} = \{x \mu_A(x) \geq \mu \}
\]
\[
\nu_A^{-\nu} = \{x \nu_A(x) \leq \nu \}
\]

\[
\mu_A^{-\nu} = \{x \mu_A(x) \geq \nu \}
\]
\[
\nu_A^{-\nu} = \{x \nu_A(x) \leq \nu \}
\]

**Definition 11.** Let $\alpha, \beta \in [0, 1]$ then we define upper ($\alpha, \beta$) cut set $A_{\alpha, \beta}$ and strong upper ($\alpha, \beta$) cut set $A_{\alpha, \beta}^*$ of $A$ as
\[
A_{\alpha, \beta} = \{x \in X, \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta \}
\]
\[
A_{\alpha, \beta}^* = \{x \in X, \mu_A(x) \geq \alpha \text{ and } \nu_A(x) < \beta \}
\]

**Definition 12.** Let $\alpha, \beta \in [0, 1]$ then we define upper ($\alpha, \beta$)-quasi cut set $A_{\alpha, \beta}^*$, strong upper ($\alpha, \beta$)-quasi cut set $A_{\alpha, \beta}^*$ of $A$ as
\[
A_{\alpha, \beta}^* = \{x \in X, \mu_A(x) + \alpha \geq 1 \text{ and } \nu_A(x) + \beta \leq 1 \}
\]
\[
A_{\alpha, \beta}^* = \{x \in X, \mu_A(x) + \alpha \geq 1 \text{ and } \nu_A(x) + \beta \leq 1 \}
\]

**Definition 13.** Let $\alpha, \beta \in [0, 1]$ then we define lower ($\alpha, \beta$)-quasi cut set $A_{\alpha, \beta}^*$, strong upper ($\alpha, \beta$)-quasi cut set $A_{\alpha, \beta}^*$ of $A$ as
\[
A_{\alpha, \beta}^* = \{x \in X, \mu_A(x) + \beta \geq 1 \text{ and } \nu_A(x) + \beta \leq 1 \}
\]
Theorem 14. If $\mu$ and $\lambda$ be two fuzzy subset of a set $X$, then

(i) $\mu \cup \lambda := \{\mu_1 \cup \lambda_1\}$
(ii) $\mu \cap \lambda := \{\mu_1 \cap \lambda_1\}$

(iii) $\mu \cup \lambda : \{x \in X \mid (\mu \cup \lambda)(x) \geq t\}$
(iv) $\mu \cap \lambda : \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$

(v) $\mu \cap \lambda : \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$
(vi) $\mu \cup \lambda : \{x \in X \mid (\mu \cup \lambda)(x) \geq t\}$

Proof.

(i) We have $\mu_1 \cup \lambda_1 = \{x \in X \mid \mu_1 \cup \lambda_1(x) \geq t\} \geq \mu_1 \cup \lambda_1$

(ii) $\mu_1 \cap \lambda_1 = \{x \in X \mid \mu_1 \cap \lambda_1(x) \leq t\} \leq \mu_1 \cap \lambda_1$

(iii) $\mu \cup \lambda = \{x \in X \mid (\mu \cup \lambda)(x) \geq t\}$

(iv) $\mu \cap \lambda = \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$

(v) $\mu \cap \lambda = \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$

(vi) $\mu \cup \lambda = \{x \in X \mid (\mu \cup \lambda)(x) \geq t\}$

(xv) Let $\mu \cap \lambda = \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$

(xvi) Let $\mu \cup \lambda = \{x \in X \mid (\mu \cup \lambda)(x) \geq t\}$

Hence $\mu \cap \lambda = \{x \in X \mid (\mu \cap \lambda)(x) \leq t\}$

Theorem 15. If $A \subseteq B$ then

(i) $A \subseteq B$

(ii) $A \subseteq B$

Proof.

(i) $A \subseteq B$

(ii) $A \subseteq B$

Hence $A \subseteq B$

(i) $x \in A \Rightarrow x \in B$

(ii) $x \in A \Rightarrow x \in B$
THEOREM 16. Let $A = \mu_A, \nu_A >$ be intuitionistic fuzzy subset of $X$ and $\alpha, \beta \in [0, 1]$ then

(i) $(A \cup B)_{(\alpha, \beta)} \subseteq A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(ii) $(A \cup B)_{(\alpha, \beta)} \supseteq A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(iii) $(A \cup B)_{(\alpha, \beta)} \subseteq A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(iv) $(A \cup B)_{(\alpha, \beta)} \supseteq A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(v) $(A \cap B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(vi) $(A \cap B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(vii) $(A')_{(\alpha, \beta)} \subseteq (A_{(\beta, \alpha)})^c$

(viii) $(A')_{(\alpha, \beta)} \subseteq (A_{(\beta, \alpha)})^c$

PROOF. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ Therefore by Theorem[13]

(i) $(A_{(\alpha, \beta)} \subseteq (A \cup B)_{(\alpha, \beta)}$ and $B_{(\alpha, \beta)} \subseteq (A \cup B)_{(\alpha, \beta)}$, and therefore $(A \cup B)_{(\alpha, \beta)} \supseteq A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(ii) $(A_{(\alpha, \beta)} \subseteq (A \cup B)_{(\alpha, \beta)}$ and $B_{(\alpha, \beta)} \subseteq (A \cup B)_{(\alpha, \beta)}$, and therefore $(A \cup B)_{(\alpha, \beta)} \supseteq A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(vi) We have $A \cap B)_{(\alpha, \beta)} = \{x \in X | (\mu_A \cap \nu_B)(x) > \alpha, (\nu_A \cup \nu_B)(x) < \beta \}$

$x \in (A \cap B)_{(\alpha, \beta)}$

$\Rightarrow (\mu_A \cap \mu_B)(x) > \alpha$ and $(\nu_A \cup \nu_B)(x) < \beta$

$\Rightarrow \min\{\mu_A(x), \mu_B(x)\} > \alpha$ and $\max\{\nu_A(x), \nu_B(x)\} < \beta$

$\Rightarrow (\mu_A(x) < \alpha, \mu_B(x) < \alpha$ and $\nu_A(x) < \beta, \nu_B(x) < \beta$)

$\Rightarrow x \in A_{(\alpha, \beta)}$ and $x \in B_{(\alpha, \beta)}$

$\Rightarrow x \in A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

Therefore $(A \cap B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(viii) $x \in (A')_{(\alpha, \beta)}$

$\Rightarrow \nu_A(x) > \alpha$ and $\mu_A(x) < \beta$

$\Rightarrow \mu_A(x) < \beta$ and $\nu_A(x) > \alpha$

$\Rightarrow \mu_A(x) \not< \beta$ and $\nu_A(x) \not< \alpha$

$\Rightarrow x \in (A_{(\beta, \alpha)})^c$

Hence $(A')_{(\alpha, \beta)} \subseteq (A_{(\beta, \alpha)})^c$

(ix) Here $\alpha \geq \gamma$ and $\beta \leq \delta$

$x \in A_{(\alpha, \beta)}$

$\Rightarrow \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$

$\Rightarrow \mu_A(x) \geq \alpha \geq \gamma$ and $\nu_A(x) \leq \beta \leq \delta$

$\Rightarrow \mu_A(x) \geq \gamma$ and $\nu_A(x) \leq \delta$

$\Rightarrow x \in A_{(\gamma, \delta)}$

Hence $A_{(\alpha, \beta)} \subseteq A_{(\gamma, \delta)}$

\[\square\]

THEOREM 17. If $A \subseteq B$ then

(i) $A_{(\alpha, \beta)} \supseteq B_{(\alpha, \beta)}$

(ii) $A_{(\alpha, \beta)} \supseteq B_{(\alpha, \beta)}$

PROOF. Same as above

\[\square\]

THEOREM 18. Let $A = \mu_A, \nu_A >$ be intuitionistic fuzzy subset of $X$ and $\alpha, \beta \in [0, 1]$ then

(i) $(A \cup B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(ii) $(A \cup B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

(iii) $(A \cap B)_{(\alpha, \beta)} \subseteq A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(iv) $(A \cap B)_{(\alpha, \beta)} \subseteq A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(v) $(A \cap B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(vi) $(A \cap B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(vii) $(A')_{(\alpha, \beta)} \subseteq (A_{(\beta, \alpha)})^c$

(viii) $(A')_{(\alpha, \beta)} \subseteq (A_{(\beta, \alpha)})^c$

PROOF. (i) Let

$x \in (A \cup B)_{(\alpha, \beta)}$

$\Leftrightarrow (\mu_A \cup \mu_B)(x) \leq \alpha \text{ and } (\nu_A \cup \nu_B)(x) \geq \beta$

$\Rightarrow \max\{\mu_A(x), \mu_B(x)\} \leq \alpha \text{ and } \min\{\nu_A(x), \nu_B(x)\} \geq \beta$

$\Rightarrow \mu_A(x) \leq \alpha, \mu_B(x) \leq \alpha \text{ and } \nu_A(x) \geq \beta, \nu_B(x) \geq \beta$

$\Rightarrow \mu_A(x) \leq \alpha, \nu_A(x) \geq \beta \text{ and } \mu_B(x) \leq \alpha, \nu_B(x) \geq \beta$

$\Rightarrow x \in A_{(\alpha, \beta)} \text{ and } x \in B_{(\alpha, \beta)}$

$\Rightarrow x \in A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

Hence $(A \cup B)_{(\alpha, \beta)} = A_{(\alpha, \beta)} \cap B_{(\alpha, \beta)}$

(iii) Let

$x \in A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)}$

$\Rightarrow x \in A_{(\alpha, \beta)} \text{ or } x \in B_{(\alpha, \beta)}$

$\Rightarrow \mu_A(x) \leq \alpha \text{ and } \nu_A(x) \geq \beta \text{ or } \mu_B(x) \leq \alpha \text{ and } \nu_B(x) \geq \beta$

$\Rightarrow \mu_A(x) \leq \alpha \text{ or } \mu_B(x) \leq \alpha \text{ and } \nu_A(x) \geq \beta \text{ or } \nu_B(x) \geq \beta$

$\Rightarrow \min\{\mu_A(x), \mu_B(x)\} \leq \alpha \text{ and } \max\{\nu_A(x), \nu_B(x)\} \geq \beta$

$\Rightarrow (\mu_A \cap \mu_B)(x) \leq \alpha \text{ and } \nu_A(x) \geq \beta \text{ or } \nu_B(x) \geq \beta$

$\Rightarrow x \in (A \cap B)_{(\alpha, \beta)}$

Hence $A_{(\alpha, \beta)} \cup B_{(\alpha, \beta)} \subseteq (A \cap B)^{(\alpha, \beta)}$

\[\square\]

THEOREM 19. If $A \subseteq B$ then

(i) $A_{(\alpha, \beta)} \subseteq B_{(\alpha, \beta)}$

(ii) $A_{(\alpha, \beta)} \subseteq B_{(\alpha, \beta)}$

PROOF. Same as above

\[\square\]
THEOREM 20. Let $A = \langle \mu_A, \nu_A \rangle >$ be intuitionistic fuzzy subset of $X$ and $\alpha, \beta \in [0, 1]$ then

(i) $(A \cup B)_{<\alpha, \beta>} \supseteq A_{<\alpha, \beta>} \cup B_{<\alpha, \beta>}$

(ii) $(A \cup B)_{<\alpha, \beta>} \supseteq A_{<\alpha, \beta>} \cup B_{<\alpha, \beta>}$

(iii) $(A \cap B)_{<\alpha, \beta>} = A_{<\alpha, \beta>} \cap B_{<\alpha, \beta>}$

(iv) $(A \cap B)_{<\alpha, \beta>} = A_{<\alpha, \beta>} \cap B_{<\alpha, \beta>}$

(v) $(A')_{<\alpha, \beta>} \subseteq (A_{<\beta, \alpha>})^c$

PROOF. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ Therefore by Theorem

(i) $A_{<\alpha, \beta>} \subseteq (A \cup B)_{<\alpha, \beta>} = A_{<\alpha, \beta>} \cup B_{<\alpha, \beta>}$

(ii) $(A \cap B)_{<\alpha, \beta>} = A_{<\alpha, \beta>} \cap B_{<\alpha, \beta>}$

PROOF. Straightforward.

THEOREM 22. Let $A = \langle \mu_A, \nu_A \rangle >$ be intuitionistic fuzzy subset of $X$ and $\alpha, \beta \in [0, 1]$ then

(i) $(A \cup B)_{<\alpha, \beta>} \subseteq A_{<\alpha, \beta>} \cup B_{<\alpha, \beta>}$

(ii) $(A \cap B)_{<\alpha, \beta>} \subseteq A_{<\alpha, \beta>} \cap B_{<\alpha, \beta>}$

PROOF. Same as above.

THEOREM 23. If $A \subseteq B$ then $A_{<\alpha, \beta>} \subseteq B_{<\alpha, \beta>}$

PROOF. Straightforward.

THEOREM 24. Let $A = \langle \mu_A, \nu_A \rangle >$ be intuitionistic fuzzy subset of $X$ and $t \in [0, 1]$, then $(A \cup B)_t = A_t \cup B_t$

PROOF. Straightforward.

THEOREM 25. Let $f : X \rightarrow Y$ be a mapping, then

(i) $f(\mu) \subseteq (f(\mu))_t$

(ii) $f(\mu) \subseteq (f(\mu))_t$

(iii) $f(\mu) \subseteq (f(\mu))_t$

(iv) $f(\mu) \subseteq (f(\mu))_t$

(v) $f(\mu) \subseteq (f(\mu))_t$

(vi) $f(\mu) \subseteq (f(\mu))_t$

(vii) $f(\mu) \subseteq (f(\mu))_t$

(viii) $f(\mu) \subseteq (f(\mu))_t$

(ix) $f(\mu) \subseteq (f(\mu))_t$

(x) $f(\mu) \subseteq (f(\mu))_t$

PROOF. (i) Let $y \in f(\mu)$ be any element, then there exists $x \in \mu$ such that $f(x) = y$ and $\mu(x) \geq t$

$\Rightarrow \forall \{x : y \in f(\mu) \subseteq \mu(x) \geq t \}

\Rightarrow f(\mu) \subseteq \mu(x) \geq t

\Rightarrow y \in f(\mu)$

Hence $f(\mu) \subseteq (f(\mu))_t$

(iv) Let $y \in f < \mu >$ be any element, then there exists $x \in \mu$ such that $f(x) = y$ and $\mu(x) + t > 1$
⇒ \forall \{x(\mu) : \mu f^{-1}(y)) + t > 1 \nRightarrow f(\mu)(t) + t > 1 \nRightarrow f(\mu)(y) + t > 1 \nRightarrow y < f(\mu) > t

Hence \( f(\mu > t) \leq f(\mu) > t \)

(vii) we have

\[(f^{-1}(\mu))_{\mu} = \{x \in X | f^{-1}(\mu)(x) > t \}
\]

\[= \{x \in X | f(x) + \mu > t \}
\]

\[= \{x \in X \mid f(x) \in \mu \}
\]

\[= \{x \in X \mid x \in f^{-1}(\mu) \}
\]

Hence \( f^{-1}(\mu) = (f^{-1}(\mu))_{\mu} \)

(ix) we have

\(< f^{-1}(\mu) > t = \{x \in X | f^{-1}(\mu)(x) + t > 1 \}
\]

\[= \{x \in X | f(x) + \mu > t \}
\]

\[= \{x \in X \mid f(x) \in \mu \}
\]

\[= \{x \in X \mid x \in f^{-1}(\mu) \}
\]

Hence \( f^{-1}(\mu) = (f^{-1}(\mu))_{\mu} \)

\[\implies y \in (f(A))_{<\alpha,\beta>}
\]

Hence \( (f(A))_{<\alpha,\beta>} \subseteq (f(A))_{<\alpha,\beta>}

(xiv) we have

\[\{f^{-1}(B)_{<\alpha,\beta>} \}
\]

\[= \{x \in [\mu f^{-1}(B)(x) + \alpha \geq 1, \nu f^{-1}(B)(x) + \beta \leq 1 \}
\]

\[= \{x \in X | f(x) \in B_{<\alpha,\beta>} \}
\]

\[= \{x \in X \mid x \in f^{-1}(B_{<\alpha,\beta>} \}
\]

Hence \( f^{-1}(B_{<\alpha,\beta>} = (f^{-1}(B))_{<\alpha,\beta>} \)

3. Cut Set of Cartesian Product of Fuzzy Sets and Intuitionistic Fuzzy Sets

Definition 27. Let \( \mu, \nu \) be two fuzzy subsets of \( X \) and \( Y \) respectively then their cartesian product of \( \mu \) and \( \nu \) is denoted by \( \mu \times \nu \) and is defined as \( \mu \times \nu = \{x \times y, (\mu \times \nu)(x, y) | x \in X, y \in Y \} \), where \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} \)

Theorem 28. Let \( \mu, \nu \) be two fuzzy subsets of \( X \) and \( Y \) respectively then

(i) \( (\mu \times \nu)(x, y) \geq t \)

(ii) \( (\mu \times \nu)_{\mu} = (\mu)_{\mu} \times (\nu)_{\mu} \)

(iii) \( (\mu \times \nu)_{\nu} = (\mu)_{\nu} \times (\nu)_{\nu} \)

(iv) \( (\mu \times \nu)_{\mu} \geq (\mu)_{\mu} \times (\nu)_{\nu} \)

Proof. (i) Let \( (x, y) \in (\mu \times \nu) \) be any element

\[\implies (\mu \times \nu)(x, y) \geq t \]

\[\implies \min \{\mu(x), \nu(y)\} \geq t \]

\[\implies \mu(x) \geq t, \nu(y) \geq t \]

\[\implies x \in (\mu)_{\mu}, y \in (\nu)_{\nu} \]

\[\implies (x, y) \in (\mu)_{\mu} \times (\nu)_{\nu} \]

Hence \( (\mu \times \nu)_{\mu} = (\mu)_{\mu} \times (\nu)_{\nu} \)

(iii) \( (\mu \times \nu)_{\nu} = (\mu)_{\nu} \times (\nu)_{\nu} \)

Let \( (x, y) \in (\mu \times \nu)_{\nu} \) be any element

\[\implies (\mu \times \nu)(x, y) \geq t \]

\[\implies \min \{\mu(x), \nu(y)\} \geq t \]

\[\implies \mu(x) \geq t, \nu(y) \geq t \]

\[\implies x \in (\mu)_{\nu}, y \in (\nu)_{\nu} \]

\[\implies (x, y) \in (\mu)_{\nu} \times (\nu)_{\nu} \]

Hence \( (\mu \times \nu)_{\nu} = (\mu)_{\nu} \times (\nu)_{\nu} \)

Definition 29. In \( \mathbb{R}^{2} \) there are six ways cartesian product of two IFs are defined, here we use only two ways viz. \( x \times y \) and \( \times \). Let \( A = (\mu_{A}, \nu_{A}) \) and \( B = (\mu_{B}, \nu_{B}) \) be any two IFs of \( X \) and \( Y \) respectively. Then their cartesian product of \( A \times B \) is defined by \( (A \times B)(x, y) = \{< (x, y), \mu_{A}(x, y), \nu_{B}(x, y) | x, y \in X \} \) where \( \mu_{A} \times B(x, y) = \min \{\mu_{A}(x), \mu_{B}(y)\} \)

and \( \nu_{A} \times B(x, y) = \max \{\nu_{A}(x), \nu_{B}(y)\} \).
\[ \mu_{(A \times B)}(x,y) = \max \{ \mu_A(x) \cdot \mu_B(y) \}, \nu_{(A \times B)}(x,y) = \min \{ \nu_A(x) \cdot \nu_B(y) \} \]

**Theorem 30.** Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be any two IFSs of \( X \) and \( Y \) respectively, then

(i) \( (A \times B)(\alpha, \beta) = A(\alpha) \times B(\beta) \)

(ii) \( (A \times B)(\alpha, \beta) = A(\alpha) \times B(\beta) \)

(iii) \( (A \times B)(\alpha, \beta) \subseteq A^{(\alpha)} \times B^{(\beta)} \)

(iv) \( (A \times B)(\alpha, \beta) \subseteq A^{(\alpha)} \times B^{(\beta)} \)

(v) \( (A \times B)(\alpha, \beta) \subseteq A^{(\alpha)} \times B^{(\beta)} \)

(vi) \( (A \times B)(\alpha, \beta) \subseteq A^{(\alpha)} \times B^{(\beta)} \)

Hence \( (A \times B)(\alpha, \beta) \subseteq A^{(\alpha)} \times B^{(\beta)} \)

\[ \text{Let } (x,y) \in (A \times B)(\alpha, \beta) \text{ be any element} \]

\[ \Rightarrow \mu_{(A \times B)}(x,y) \geq \alpha \quad \text{and} \quad \nu_{(A \times B)}(x,y) \leq \beta \]

\[ \Rightarrow \max \{ \mu_A(x) \cdot \mu_B(y) \} \geq \alpha \quad \text{and} \quad \min \{ \nu_A(x) \cdot \nu_B(y) \} \leq \beta \]

\[ \Rightarrow \mu_A(x) \geq \alpha \quad \text{or} \quad \mu_B(y) \geq \alpha \quad \text{and} \quad \nu_A(x) \leq \beta \quad \text{or} \quad \nu_B(y) \leq \beta \]

\[ \Rightarrow \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \quad \text{and} \quad \mu_B(y) \geq \alpha, \nu_B(y) \leq \beta \]

\[ \Rightarrow x \in A^{(\alpha)} \quad \text{and} \quad y \in B^{(\beta)} \]

\[ \Rightarrow (x,y) \in A^{(\alpha)} \times B^{(\beta)} \]

\[ \square \]

4. CONCLUSION

In this paper, we have discussed detail theory of cut sets in fuzzy sets and in intuitionistic fuzzy sets. It is observed that the papers [1], [6], and [7] are purely based on cut sets. Now our expectation is that this work will build foundations for further study of the theory of cut sets in both fuzzy sets and intuitionistic fuzzy sets. Also, in our opinion, the definition of various types of cut sets can be extended to cut sets of interval-valued fuzzy sets and cubic fuzzy sets.

5. REFERENCES


