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Some Structural Properties of Unitary Addition Cayley Graphs

Naveen Palanivel
Department of Mathematics,
National Institute of Technology,
Calicut, Kerala,
India-673601

Chithra.A.V
Department of Mathematics,
National Institute of Technology,
Calicut, Kerala,
India-673601

ABSTRACT

For a positive integer $n > 1$, the *unitary addition Cayley graph* G_n is the graph whose vertex set is $V(G_n) = Z_n = \{0, 1, 2, \cdots, n-1\}$ and the edge set $E(G_n) = \{ab \mid a, b \in Z_n, a + b \in U_n\}$ where $U_n = \{a \in Z_n \mid gcd(a, n) = 1\}$. For G_n the independence number, chromatic number, edge chromatic number, diameter, vertex connectivity, edge connectivity and perfectness are determined.

Keywords

Unitary Cayley Graph, Unitary Addition Cayley Graph, Chromatic Number, Independence Number, Connectivity, Perfectness.

1. INTRODUCTION

Throughout this paper, we consider only finite, simple, undirected graphs. For standard terminology and notation in graph theory we follow [8] and algebraic graph theory we follow [1], [7]. *Degree of a vertex* v in a graph G is the number of edges incident with that vertex and it is denoted by $d(v)$. $\delta(G)$ denotes *minimum degree* of the graph G and $\Delta(G)$ denotes *maximum degree* of the graph G. The *vertex connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph and the *edge connectivity* $\lambda(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected or trivial graph. A graph is called *regular* if all vertices have same degree and a graph is called $(r_1, r_2)-$ *semi regular* if its vertex set can be partitioned into two subsets V_1 and V_2 such that all the vertices in V_i are of degree r_i for $i = 1, 2$.
A shortest $u - v$ path is called a *geodesic*. The *diameter* of a connected graph is the length of any longest geodesic. The set of vertices in a graph is independent if no two of them are adjacent. The largest number of vertices in such a set is called the *independence number* of G and it is denoted by $\beta_0(G)$. An independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the *mactching number* $\beta_1(G)$ or β_1. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices which covers all the edges of a graph G is called a *vertex cover* for G, while a set of edges which covers all the vertices is an *edge cover*. The minimum number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha_0(G)$. $\alpha_1(G)$ is the smallest number of edges in any edge cover of G and is called its *edge covering number*.
A *clique* of a graph G is a complete sub graph of G, and the clique of largest possible size is referred to as a maximum clique. The *clique number* of a graph G is the number of vertices in a maximum clique of G, denoted $\omega(G)$. The *vertex chromatic number* $\chi(G)$ is defined as the minimum number of colours such that no two adjacent vertices share a common colour. The *edge chromatic number* $\chi'(G)$ is the minimum number of colours such that no two adjacent edges share a common colour.
A graph G is *perfect*, if for every induced sub graph $G' \subseteq G$ the clique number and the chromatic number coincide, $\omega(G') = \chi(G')$.

Let Γ be a multiplicative group with identity 1. For $S \subseteq \Gamma, 1 \notin S, S^{-1} = \{s^{-1} \mid s \in S\} = S$ the *Cayley graph* $X = Cay(\Gamma, S)$ is the undirected graph having vertex set $V(X) = \Gamma$ and edge set $E(X) = \{(a, b) \mid ab^{-1} \in S\}$. The cayley graph X is regular of degree $|S|$.

For a positive integer $n > 1$, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a - b \in U_n$. The unitary Cayley graph X_n is also defined as, $X_n = Cay(Z_n, U_n)$. The graph X_n is regular of degree $|U_n| = \phi(n)$, where $\phi(n)$ denotes the Euler phi function [5].

For a positive integer $n > 1$, the *unitary addition Cayley graph* $G_n = Cay^+(Z_n, U_n)$ is the graph whose vertex set is $Z_n = \{0, 1, 2, \cdots, n-1\}$ and the edge set $E(G_n) = \{ab \mid a, b \in Z_n, a + b \in U_n\}$ where $U_n = \{a \in Z_n \mid gcd(a, n) = 1\}$. The graph G_n is *regular* if n is even and *semi regular* if n is odd [12].

Figures 1 and 2 show some examples of unitary addition Cayley graphs.

2. PRELIMINARIES

THEOREM 1 [8]. *The minimum number of vertices separating two nonadjacent vertices s and t is the maximum number of disjoint $s - t$ paths.*

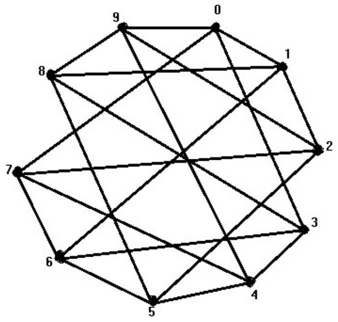

Fig. 1. G_{10}

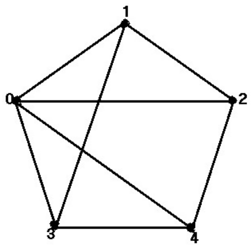

Fig. 2. G_5

THEOREM 2 [8]. *For any graph G, the edge chromatic number satisfies the inequalities,* $\Delta \leq \chi'(G) \leq \Delta + 1$.

THEOREM 3 [12]. *The unitary addition Cayley graph G_n is isomorphic to the unitary Cayley graph X_n if and only if n is even.*

THEOREM 4 [2]. *The edge chromatic number $\chi'(X_n)$ of the unitary Cayley graph X_n is $\phi(n)$ if n is even.*

THEOREM 5 [2]. *The edge connectivity $\lambda(X_n)$ of the unitary Cayley graph X_n is $\phi(n)$ if n is even.*

THEOREM 6 [9]. *The unitary Cayley graph X_n has vertex connectivity $\kappa(X_n) = \phi(n)$.*

THEOREM 7 [9]. *If p is the smallest prime divisor of n, then we have $\chi(X_n) = \omega(X_n) = p$.*

THEOREM 8 [12]. *Let m be any vertex of the unitary addition cayley graph G_n. Then*

$$d(m) = \begin{cases} \phi(n) & \text{if } n \text{ is even,} \\ \phi(n) & \text{if } n \text{ is odd and } gcd(m,n) \neq 1, \\ \phi(n) - 1 & \text{if } n \text{ is odd and } gcd(m,n) = 1. \end{cases}$$

THEOREM 9 [10]. *Let p be a prime number. Then $x^2 \equiv 1(modp)$ if and only if $x \equiv \pm 1(modp)$.*

THEOREM 10 [6]. *The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element.*

COROLLARY 1 [12]. *The total number of edges in the unitary addition Cayley graph G_n is*

$$|E(G_n)| = \begin{cases} \frac{1}{2} n\phi(n) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)\phi(n) & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 11 [11]. *Let G be a graph with diameter ≤ 2. Then the edge connectivity $\lambda(G)$ is equal to the minimum degree $\delta(G)$.*

THEOREM 12 [4]. ***Strong Perfect Graph Theorem(SPGT).*** *A graph G is perfect if and only if G and its complement $\overline{G}$ have no induced cycles of odd length atleast 5.*

THEOREM 13 [3]. *Let $G \neq K_n$ be a graph of order n, then $\kappa(G) \geq 2\delta(G) + 2 - n$.*

OBSERVATION 1. *Unitary addition cayley graph $G_n (n \geq 3)$ can be decomposed into $\frac{\phi(n)}{2}$ disjoint Hamiltonian cycles if n is even and can be decomposed into $\frac{\phi(n)}{2} - 1$ disjoint Hamiltonian cycles if n is odd.*

3. CONNECTIVITY AND INDEPENDENCE OF UNITARY ADDITION CAYLEY GRAPH

LEMMA 14. *If n is odd then the number of elements in U_n of order 2 is 2^r(we consider identity 1 has order 2) and these elements are represented in the form $H = \{x \in U_n \mid x = \beta_x Z\}$ where $\beta_x = \begin{bmatrix} a_{1x} & a_{2x} & a_{3x} & \cdots & a_{rx} \end{bmatrix}$, $Z = \begin{bmatrix} (Z_1)^{\left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)} \\ (Z_2)^{\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right)} \\ \vdots \\ (Z_r)^{\left(p_r^{\alpha_r} - p_r^{\alpha_r - 1}\right)} \end{bmatrix}$, $Z_i = n/p_i^{\alpha_i}$ and $a_{ix} \in \{1, -1\}, 1 \leq i \leq r$, where r is the number of distinct prime factors of n.*

PROOF. If m and n are relatively prime then U_{mn} is isomorphic to $U_m \oplus U_n$. Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$. Then each pair of elements $\left(p_i^{\alpha_i}, p_j^{\alpha_j}\right), i \neq j$, is relatively prime and $U_n = U_{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}} \approx U_{p_1^{\alpha_1}} \oplus U_{p_2^{\alpha_2}} \oplus \cdots \oplus U_{p_r^{\alpha_r}}$.
The number of elements in U_n of order 2 is 2^r, since the order of an element of a direct product of a finite number of finite groups is the least common multiple of the order of the components of the element.
Let $x^2 \equiv 1(modn)$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$
This implies $x^2 \equiv 1(modp_1^{\alpha_1})$
$x^2 \equiv 1(modp_2^{\alpha_2})$
$\vdots$
$x^2 \equiv 1(modp_r^{\alpha_r})$
This implies $x \equiv \pm 1(modp_1^{\alpha_1})$
$x \equiv \pm 1(modp_2^{\alpha_2})$
$\vdots$
$x \equiv \pm 1(modp_r^{\alpha_r})$
Using Chinese remainder theorem and Eulerś theorem, we get $x = \pm (Z_1)^{\left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)} \pm (Z_2)^{\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right)} \pm \cdots \pm (Z_r)^{\left(p_r^{\alpha_r} - p_r^{\alpha_r - 1}\right)} (modn)$ where $Z_i = n/p_i^{\alpha_i}, 1 \leq i \leq r$.
$x = \beta_x Z(modn)$ where $\beta_x = \begin{bmatrix} a_{1x} & a_{2x} & a_{3x} & \cdots & a_{rx} \end{bmatrix}$ and $Z = \begin{bmatrix} (Z_1)^{\left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)} \\ (Z_2)^{\left(p_2^{\alpha_2} - p_2^{\alpha_2 - 1}\right)} \\ \vdots \\ (Z_r)^{\left(p_r^{\alpha_r} - p_r^{\alpha_r - 1}\right)} \end{bmatrix}$, $Z_i = n/p_i^{\alpha_i}$ and $a_{ix} \in \{1, -1\}, 1 \leq i \leq r$. □

THEOREM 15. *Let n be an odd number. Then the unitary addition Cayley graph G_n is k− partite, $k = \frac{\phi(n)}{2^r} + r$, where r is the number of distinct prime factors of n.*

PROOF. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, $p_1 < p_2 < \cdots < p_r$, p_i's are distinct prime factors of n, $1 \leq i \leq r$. Then the sets $\langle p_1 \rangle, \langle p_i \rangle \backslash \cup_{1 \leq j \leq i-1} \{\langle p_j \rangle\}, 2 \leq i \leq r$ are distinct independent sets in G_n. So $V(G_n) - U_n$ splitting into r distinct independent sets.
By lemma 14, the number of elements of order 2 in U_n is 2^r and they are $H = \{x \in U_n \mid x = \beta_x Z\}$.
Suppose $x, y \in H$, then $x = \beta_x Z, y = \beta_y Z$ and $x + y = (\beta_x + \beta_y)Z$, so $\beta_x + \beta_y = [b_1 b_2 \cdots b_r]$, $b_i = a_{ix} + a_{iy} \in \{0, 2, -2\}$, $1 \leq i \leq r$.
If all b_i's are zeros then $gcd(x+y, n) = gcd(0, n) = n$, it implies that $x + y \notin U_n$. If some b_i's are non-zeros, say $b_{s_1}, b_{s_2}, \cdots, b_{s_t}$, $1 \leq t \leq r-1$ then corresponding $(Z_{s_l})^{(p_{s_l}^{\alpha_{s_l}} - p_{s_l}^{\alpha_{s_l}-1})}$ does not contain $p_{s_l}^{\alpha_{s_l}}$, $1 \leq l \leq r-1$, therefore $gcd(x+y, n) = \frac{n}{q}$ where $q = p_{s_1}^{\alpha_{s_1}} p_{s_2}^{\alpha_{s_2}} \cdots p_{s_t}^{\alpha_{s_t}}$. It implies that $x + y \notin U_n$.
That is H is an independent set in G_n.
In a similar manner, we can prove that $2^l H$ is an independent set in G_n, where $1 \leq l < \frac{\phi(n)}{2^r}$.
Suppose $x \in 2^l H$
$\Leftrightarrow x = 2^l \beta_x Z = (2^l \beta_x) Z$
$\Leftrightarrow x = \beta_k Z$ where $\beta_k = 2^l \beta_x$ and it has elements $+2^l$ and -2^l
$\Leftrightarrow x \notin 2^t H, 0 \leq l, t < \frac{\phi(n)}{2^r}, l \neq t$
So $H, 2^1 H, 2^2 H, \cdots, 2^l H$ are distinct independent sets, each set has 2^r elements.
Hence the unitary addition Cayley graph G_n is k partite, $k = \frac{\phi(n)}{2^r} + r$, where r is the number of distinct prime factors of n. □

THEOREM 16. *Independence number of the unitary addition Cayley graph G_n, $n \geq 3$, is*

$$\beta_0(G_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n}{p_1} & \text{if } n \text{ is odd but not a prime number}, \\ 2 & \text{if } n \text{ is prime}. \end{cases}$$

. *where* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, p_1 < p_2 < \cdots < p_r, \alpha_i \geq 1, 1 \leq i \leq r$.

PROOF. Suppose n is even. Then the sets $H = \{0, 2, 4, \cdots, n-2\}$ and $L = \{1, 3, 5, \cdots, n-1\}$ are independent in G_n. It implies G_n has only two independent sets and both has $\frac{n}{2}$ elements. Hence independence number is $\frac{n}{2}$.
Next, suppose n is odd, but not a prime. Then the sets $K_i = < p_i >, 1 \leq i \leq r$, are independent in G_n. In these sets K_1 is maximum, since number of elements in K_i are $\frac{n}{p_i}$.
Any independent set in U_n has atmost 2^r elements where r is the number of distinct prime factors of n, but $\frac{n}{p_1} > 2^r$. So K_1 is a maximum independent set and hence independence number is $\frac{n}{p_1}$.
Suppose $n = p$, where p is a prime number, then the vertex zero has degree $p - 1$ and all other vertices have degree $p - 2$. Let $W = V(G_n) - \{0\}$. For every $v \in W$,v is adjacent to all vertices in G_n except the vertex $n - v$ in W, so the independence number is 2. □

COROLLARY 2. *Covering number of the unitary addition Cayley graph G_n, $n \geq 3$,$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, p_1 < p_2 < \cdots < p_r, \alpha_i \geq 1, 1 \leq i \leq r$ is*

$$\alpha_0(G_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ (\frac{p_1 - 1}{p_1})n & \text{if } n \text{ is odd but not a prime number}, \\ n - 2 & \text{if } n \text{ is prime}. \end{cases}$$

THEOREM 17. *Matching number of the unitary addition Cayley graph G_n, $n \geq 3$, is*

$$\beta_1(G_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

PROOF. In G_n, the generating set U_n must contain 1.
Suppose n is even. Then edge set $E_1 = \{(0, 1), (n-1, 2), (n-2, 3), \cdots, (\frac{n+2}{2}, \frac{n}{2})\}$ is an independent set in G_n and $|E_1| = \frac{n}{2}$.
Suppose the matching number is greater than $\frac{n}{2}$, by definition of matching number the number of end vertices are greater than $2(\frac{n}{2})$. It contradicts the total number of vertices in G_n. So matching number is $\frac{n}{2}$.
Suppose n is odd, then the edge set $E_2 = \{(0, 1), (n-1, 2), (n-2, 3), \cdots, (\frac{n+3}{2}, \frac{n-1}{2})\}$ is an independent set in G_n and $|E_2| = \frac{n-1}{2}$.
Suppose the matching number is greater than $\frac{n-1}{2}$, that is matching number is greater than or equal to $\frac{n+1}{2}$. By definition of matching number the number of end vertices are greater than or equal to $2(\frac{n+1}{2})$. It contradicts the total number of vertices in G_n. So matching number is $\frac{n-1}{2}$. □

COROLLARY 3. *An edge covering number of the unitary addition Cayley graph G_n, $n \geq 3$, is*

$$\alpha_1(G_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n+1}{2} & \text{if } n \text{ is odd }. \end{cases}$$

OBSERVATION 2. *Let $u, v, w \in V(G_n)$. Vertex w is a common neighbour of u and v, if and only if $gcd(u + w, n) = gcd(v + w, n) = 1$. Then there exist unique $x, y \in Z_n$ such that*
$u + w \equiv x \mod n, v + w \equiv y \mod n$.
Now $w \equiv x - u \equiv y - v$ becomes a common neighbour of u and v, if and only if $x - y \equiv u - v \mod n$, $x, y \in U_n$. This congruence has atleast one solution if n is odd.

THEOREM 18. *The diameter of the unitary addition cayley graph G_n, $n > 2$, is*

$$diam(G_n) = \begin{cases} 2 & \text{if } n \text{ is prime}, \\ 2 & \text{if } n \text{ is even and } n = 2^m,\ m \geq 2, \\ 3 & \text{if } n \text{ is even and } n \neq 2^m, m \geq 2, \\ 2 & \text{if } n \text{ is odd but not a prime }. \end{cases}$$

PROOF. Suppose $n = p$, p is prime, then $U_p = \{1, 2, 3, \cdots, p-1\}$. If $u \in U_p$ then u is adjacent to $p - 2$ vertices including 0 and 0 is adjacent to all vertices. This implies diameter of G_n is 2.
Suppose n is even and $n = 2^m (m \geq 2)$, then $U_n = \{1, 3, 5, \cdots, n-1\}$. An element 0 in $V(G_n)$ is adjacent to a vertex u where $u \in U_n$ and u is adjacent to all even vertices. This implies diameter of G_n is 2.
Suppose $n (n \neq 2^m, m \geq 2)$ is even and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $p_1 < p_2 \cdots < p_r$, $\alpha_i \geq 1$, p_i are distinct prime factors of n, $1 \leq i \leq r$.
In G_n, zero is non adjacent to p_i, $1 \leq i \leq r$, also zero and $p_i (2 \leq i \leq r)$ has no common vertex, since zero is adjacent to some odd vertices and $p_i (2 \leq i \leq r)$ is adjacent to some even vertices. Therefore $diam(G_n) \geq 3$.
In G_n both even or both odd vertices are non adjacent. If u and v are odd (even) vertices in G_n then they have atleast one common vertex w in G_n and w is even (odd), since G_n is connected. We consider two non adjacent vertices v(even) and u(odd) in G_n, v is adjacent to some vertex x(odd) in G_n. Here x and u are odd vertices then they have a common vertex y(even) in G_n. Passing along

v, x, y and u, shows $diam(G_n) = d(v, u) \leq 3$.
Suppose n is odd but not a prime, then every pair of distinct non adjacent vertices have a common neighbour. This implies diameter of G_n is 2. □

THEOREM 19. *Edge connectivity of the unitary addition Cayley graph G_n is*

$$\lambda(G_n) = \begin{cases} \phi(n) & \text{if } n \text{ is even}, \\ \phi(n) - 1 & \text{if } n \text{ is odd}. \end{cases}$$

PROOF. Suppose n is odd $\lambda(G_n) = \phi(n) - 1$, by Theorems 8, 11 and 18.
Suppose n is even $\lambda(G_n) = \phi(n)$, by Theorems 3 and 5. □

COROLLARY 4. *Vertex connectivity of the unitary addition Cayley graph G_n is*

$$\kappa(G_n) = \begin{cases} \phi(n) & \text{if } n \text{ is even}, \\ \phi(n) - 1 & \text{if } n \text{ is prime} \end{cases}$$

PROOF. Suppose n is even $\kappa(G_n) = \phi(n)$, by Theorems 3 and 6.
Suppose n is prime say p, then $\kappa(G_p) = \phi(p) - 1$, by Theorems 13 and 19. □

REMARK 1. *For all n, $2\phi(n) - n \leq \kappa(G_n) \leq \phi(n) - 1$.*

4. CHROMATIC AND CLIQUE NUMBER OF UNITARY ADDITION CAYLEY GRAPH

THEOREM 20. *Chromatic number of the unitary addition Cayley graph G_n is $\chi(G_n) = 2$ if n is even and $\chi(G_n) \leq \frac{\phi(n)}{2^r} + r$ if n is odd, where r is the number of distinct prime factors of n.*

PROOF. Suppose n is even. By Theorems 3 and 7, $\chi(G_n) = 2$. If n is odd then G_n splitting into $\frac{\phi(n)}{2^r} + r$ distinct independent sets. Therefore $\chi(G_n) \leq \frac{\phi(n)}{2^r} + r$. □

THEOREM 21. *Edge chromatic number of the unitary addition Cayley graph G_n is $\phi(n)$.*

PROOF. Suppose n is odd. From the definition of proper edge colouring G_n contains atmost $\frac{n-1}{2}$ edges of a same colour. By Corollary 1, atmost $\phi(n)$ colours are needed to colour G_n. So $\chi'(G_n) \leq \phi(n)$.
By Theorem 2, $\phi(n) \leq \chi'(G_n)$.
Therefore $\chi'(G_n) = \phi(n)$.
Suppose n is even. By Theorem 3 and Theorem 4, edge chromatic number is $\phi(n)$. □

THEOREM 22. *Clique number of the unitary addition Cayley graph G_n is*

$$\omega(G_n) = \begin{cases} 2 & \text{if } n \text{ is even}, \\ \frac{\phi(n)}{2} + 1 & \text{if } n = p^m,\ p \neq 2 \text{ and } m \geq 2. \end{cases}$$

and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $p_1 < p_2 < \cdots < p_r$

$$\omega(G_n) \geq \begin{cases} 3 & \text{if } p_1 = 3, \\ \frac{p_1+1}{2} & \text{if } p_1 > 3. \end{cases}$$

PROOF. Case 1. Suppose n is even. By Theorems 3 and 7, $\omega(G_n) = 2$.
Case 2. For $n = p^m, p \neq 2$ and $m \geq 2$.
Let $U_n = \{\pm u_1, \pm u_2, \cdots, \pm u_k\}$.
If $\phi(n) = 2k$ and k is even, then $A = \{0, u_1, u_3, \cdots, u_{k-1}, -u_k, -u_{k-2}, \cdots, -u_2\}$ is a clique in G_n .
If $\phi(n) = 2k$ and k is odd, then $B = \{0, u_1, u_3, \cdots, u_k, -u_{k-1}, -u_{k-3}, \cdots, -u_2\}$ is a clique in G_n .
In both case $|A| = |B| = \frac{\phi(n)}{2} + 1$. So $\omega(G_n) \geq \frac{\phi(n)}{2} + 1$.
From Theorem 20 we get $\omega(G_n) \leq \frac{\phi(n)}{2} + 1$. Therefore $\omega(G_n) = \frac{\phi(n)}{2} + 1$.
Case 3. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and $p_1 = 3$.
The set $\{0, p_1, p_2\}$ is a clique in G_n, so $\omega(G_n) \geq 3$.
Case 4. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and $p_1 > 3$.
The set $\{0, 1, 2, \cdots, \frac{p_1-1}{2}\}$ is a clique in G_n, so $\omega(G_n) \geq \frac{p_1+1}{2}$. □

5. PERFECTNESS

LEMMA 23. *If n is odd and has atleast two different prime divisors, then $\overline{G}_n$ contains an induced cycle C_5 of length 5.*

PROOF. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $p_1 < p_2 < \cdots < p_r$, where r is the number of distinct prime factor of n.
Choose the vertices v_1, v_2, v_3, v_4 and v_5 in the following manner $v_1 = 0, v_2 = p_r, v_3 = p_1 p_2 \cdots p_{r-1} - p_r, v_4 = -2p_1 p_2 \cdots p_{r-1} + p_r$,
$v_5 = 2p_1 p_2 \cdots p_{r-1}$. The vertices v_1, v_2, v_3, v_4 and v_5 are distinct. These vertices form a cycle C_5 of $\overline{G}_n$, because
$v_1 + v_2 \equiv v_4 + v_5 \equiv 0 (mod p_r)$
$v_1 + v_5 \equiv v_2 + v_3 \equiv v_3 + v_4 \equiv 0 (mod p_i)$, $i = 1, 2, \cdots, r - 1$.
It follows that the edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots \{v_5, v_1\}$ belong to $\overline{G}_n$.
Next to show that this C_5 has no chords in $\overline{G}_n$.

$$\left.\begin{aligned} v_1 + v_3 &= p_1 p_2 \cdots p_{r-1} - p_r \\ v_1 + v_4 &= -2p_1 p_2 \cdots p_{r-1} + p_r \\ v_2 + v_4 &= -2(p_1 p_2 \cdots p_{r-1} - p_r) \\ v_2 + v_5 &= 2p_1 p_2 \cdots p_{r-1} + p_r \\ v_3 + v_5 &= 3p_1 p_2 \cdots p_{r-1} - p_r \end{aligned}\right\} \quad \ldots \qquad (1)$$

From (1), we get $v_1 + v_3, v_1 + v_4, v_2 + v_4, v_2 + v_5$ and $v_3 + v_5$ are non divisible by $p_i, i = 1, 2, \cdots, r$. Therefore the cycle C_5 is an induced cycle in $\overline{G}_n$. □

REMARK 2. *Unitary addition Cayley graph G_n is not perfect if n is odd and has atleast two different prime divisors.*

LEMMA 24. *Let $n = p^m$, where p is a prime number and $p > 2$. Then $\overline{G}_n$ has no induced odd cycle $C_{2k+1}, k \geq 2$.*

PROOF. Assume that $\overline{G}_n$ contains an induced cycle $C_{2k+1}, k \geq 2$, which runs through the vertices $v_1, v_2, \cdots, v_{2k+1}$ in this order.
We consider three consecutive edges $\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}, \{v_{i+2}, v_{i+3}\}$ in C_{2k+1}. This implies that $v_i + v_{i+1}, v_{i+1} + v_{i+2}$ and $v_{i+2} + v_{i+3}$ are divisible by p in $\overline{G}_n$.
Adding first and third term, we get $v_i + v_{i+1} + v_{i+2} + v_{i+3}$, which is divisible by p.
This implies that $v_i + v_{i+3}$ is divisible by p in $\overline{G}_n$.
It follows that $\{v_i, v_{i+3}\}$ is an edge in G_n. It is a contradiction to our assumption. □

LEMMA 25. *Let $n = p^m$, where p is a prime number and $p > 2$. Then G_n has no induced odd cycle $C_{2k+1}, k \geq 2$.*

PROOF. Assume that G_n contains an induced cycle $C_{2k+1}, k \geq 2$, which runs through the vertices $v_1, v_2, \cdots, v_{2k+1}$ in this order. We consider two cases.
Case 1. Atleast one $v_i \in \{< p >\}$, say v. Then v is non adjacent to all vertices $v_j \in \{< p >\}$,$1 \leq j \leq 2k+1$. It is a contradiction to our assumption.
Case 2. Let $k \geq 3$ and all $v_i \notin \{< p >\}$.
If $x \in U(p^m)$ then any vertex y non adjacent to x is of the form $y = lp - x \in U(p^m)$, $1 \leq l \leq \frac{n}{p}$. In C_{2k+1}, v_1 is non adjacent to atleast three vertices, say v_x, v_y and v_z. So $v_x = l_1p - v_1, v_y = l_2p - v_1$ and $v_z = l_3p - v_1$,$1 \leq l_1, l_2, l_3 \leq \frac{n}{p}$. Here $v_x + v_y = (l_1 + l_2)p - 2v_1$, $v_x + v_z = (l_1 + l_3)p - 2v_1$, and $v_y + v_z = (l_2 + l_3)p - 2v_1$. So $v_x + v_y, v_x + v_z, v_y + v_z \in U_n$. It is a contradiction to our assumption.
Assume that G_n contains an induced cycle C_5, which run through the vertices v_1, v_2, v_3, v_4 and v_5. So $v_1+v_4, v_1+v_3, v_2+v_4, v_2+v_5$ and $v_3 + v_5$ are divisible by p. Adding $v_1 + v_3$ and $v_2 + v_5$, we get $v_1 + v_3 + v_2 + v_5$ is divisible by p. Also $v_3 + v_5$ is divisible by p. This implies that $v_1 + v_2$ is divisible by p. It is a contradiction to our assumption. □

Combining the lemmas 24, 25 and using the property of bipartite, now we can prove the following result.

THEOREM 26. *The unitary addition Cayley graph G_n, $n \geq 2$, is perfect if and only if n is even or $n = p^m, m \geq 1$.*

6. CONCLUSION

In this paper we determine some structural properties of unitary addition Cayley graph G_n, including diameter, connectivity and perfectness.

7. REFERENCES

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