On Exponential Fuzzy Measures of Information and Discrimination

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ABSTRACT

In the present communication, two exponential fuzzy information measures are introduced and characterized axiomatically. To show the effectiveness of the proposed measure, it is compared with the existing measures. Two fuzzy discrimination and symmetric discrimination measures are defined and their validity are checked. Important properties of new measures are studied. Their applications in pattern recognition and diagnosis problem of crop disease are discussed.

Keywords:

Fuzzy set, Fuzzy information, Discrimination measure, Pattern recognition

1. INTRODUCTION

Uncertainty is natural behaviour in real systems. Probability has been traditionally used in modeling uncertainty. Since [24] instigated the idea of fuzzy sets, fuzziness becomes another way to model uncertainty. On the other hand, measuring the fuzziness of fuzzy sets is an important step in fuzzy systems. Measures of fuzziness by contrast to fuzzy measures try to indicate the degree of fuzziness of a fuzzy set. The entropy of fuzzy sets is a measure of fuzziness between fuzzy sets. [2] first introduced the axiom construction for entropy of fuzzy sets with reference to Shannon's probability entropy. [3] suggested five properties for a entropy of fuzzy sets to satisfy. [16]− [17] proposed higher order and exponential entropies. More surveys on measuring fuzziness were given by [18] and [15]. Recently, [13] considered new defined fuzzy variables and discussed about entropies on fuzzy variables. In [12], they first proposed the notion of credibility measure and then considered entropy of credibility distributions for fuzzy variables.

A perception-based theory of probabilistic reasoning with imprecise probabilities has also been explained by [25]. Some work related with uncertainty management for intelligence analysis was reported by [23] whereas the generalized information theory, its aims, results and some open problems were discussed by [8]. The fuzzy information measures have found wide applications to Engineering, Fuzzy traffic control, Fuzzy aircraft control, Computer sciences, management and Decision making, etc.

Many researchers have studied various generalized fuzzy information measures. [4] generalized Fuzzy information measure and introduced R-norm fuzzy information measure. [7] defined fuzzy entropy by combining the concepts of [22] and [18]− [19]. [6] also characterized [2] entropy as generalized Fuzzy information measure and R-norm fuzzy information measure. [5] characterized the measures of fuzzy information analogous to the sub additive information measures due to [20]. [21] generalized exponential fuzzy information measure and studied essential and some other properties. Later on [14] defined new measure of weighted fuzzy information, the findings of which have been applied to study the principle of maximum weighted fuzzy information. [27] developed trigonometric and tangent inverse trigonometric fuzzy information measures and applied these measures in strategic decision making. [28] proposed the fuzzy mean code word lengths of degree β and type (α, β) and discussed the behaviour of the proposed fuzzy mean code word lengths.

In the present paper, we introduce two new exponential fuzzy information measures and study their validity in section 2. In section 3, some comparisons are made with some existing information measures to show the effectiveness of the proposed one. Two exponential fuzzy discrimination measures are defined and the properties of these exponential fuzzy discrimination measures are studied in section 4. In section 5, the applications of these exponential fuzzy symmetric discrimination measures to the pattern recognition and diagnosis of crop disease are discussed.

2. NEW EXPONENTIAL FUZZY INFORMATION MEASURES

The following notations are used in this section. $R^+ = [0,\ \infty[$, let $FS(X)$ be the set of all fuzzy sets on a universal set X and $\mathrm{P}\,(X)$ be the set of all crisp sets on the universal set X. $\mu_A(x)$ is the membership function of $A \in FS(X)$, $[a]$ is the fuzzy set of X for which $\mu_{[a]}(x) = a,\ \forall\, x \in X\ (a \in [0,1])$. For fuzzy set A, we use A^c to express the complement of $A,\ i.e.\ \mu_{A^c}(x) = 1 - \mu_A(x),\ \forall\, x \in X$. For two fuzzy sets A and B, $A \cup B$, the union of A and B is defined as $\mu_{A\cup B}(x) = \max\{\mu_A(x),\ \mu_B(x)\}$,$A \cap B$, the intersection of A and B is defined as $\mu_{A\cap B}(x) = \min\{\mu_A(x),\ \mu_B(x)\}$. A fuzzy set A^* is called a sharpening of A, if $\mu_{A^*}(x) \geq \mu_A(x)$ when $\mu_A(x) \geq \frac{1}{2}$ and $\mu_{A^*}(x) \leq \mu_A(x)$ when $\mu_A(x) < \frac{1}{2}$.

Definition 2.1 ([24]). A fuzzy set A defined in a finite universe of discourse $X = \{x_1,\ x_2,\ ...,\ x_n\}$ is given by

$$A = \{(x_i, \mu_A(x_i)) : \mu_A(x_i) \in [0,1]\,; \forall x_i \in X\},$$

where $\mu_A(x_i)$ represents the degree of membership and is defined as

$$\mu_A(x_i) = \begin{cases} 0, \text{if } x_i \notin A \text{ and there is no ambiguity}, \\ 1, \text{if } x_i \in A \text{ and there is no ambiguity}, \\ 0.5, \text{there is max ambiguity whether } x_i \notin A \text{or } x_i \in A. \end{cases}$$

[2] first gave axioms for information measure of fuzzy sets as follows:

Definition 2.2 ([2]). A real function $H : FS(X) \to R^+$ is called fuzzy information measure on $FS(X)$, if H have the following properties:

(1) $H(D) = 0, \forall D \in P(X)$;
(2) $H\left(\left[\frac{1}{2}\right]\right) = \max_{A \in FS(X)} H(A)$;
(3) $H(A^*) \leq H(A)$, for any sharpening A^* of A;
(4) $H(A) = H(A^c), \ \forall A \in FS(X)$.

In this section, we introduce the following two new exponential fuzzy information measures:

$$H(A) = \sum_{i=1}^{n} \left[e - \mu_A(x_i)e^{\mu_A(x_i)} - (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}\right] \tag{1}$$

$$H_\alpha(A) = \sum_{i=1}^{n} \left[e^\alpha - \mu_A(x_i)e^{\alpha\mu_A(x_i)} - (1 - \mu_A(x_i))e^{\alpha(1-\mu_A(x_i))}\right], \tag{2}$$

where $\alpha > 0, \alpha \neq 1$.

Theorem 1. The measure (1) is a valid measure of fuzzy information.

Proof: To prove that the measure (1) is a valid fuzzy information measure, we show that four postulate (P1) to (P4) hold.

(P1) (Crispness): If $H(A) = 0$ then

$$\sum_{i=1}^{n} \left[e - \mu_A(x_i)e^{\mu_A(x_i)} - (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}\right] = 0$$

Either $\mu_A(x_i) = 0$ or $1 \ \forall i = 1, 2, ..., n$.
A is a crisp set.
Conversely, let A be a crisp set, then either $\mu_A(x_i) = 0$ or $1 \ \forall i = 1, 2, ..., n$.
It implies $\left[e - \mu_A(x_i)e^{\mu_A(x_i)} - (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}\right] = 0$

$$\sum_{i=1}^{n} \left[e - \mu_A(x_i)e^{\mu_A(x_i)} - (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}\right] = 0$$

$$H(A) = 0.$$

Hence $H(A) = 0$ if and only if A is non-fuzzy set or crisp set.

(P2) (Maximality): Differentiating $H(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{dH(A)}{d\mu_A(x_i)} = \sum_{i=1}^{n} [-e^{\mu_A(x_i)} - \mu_A(x_i)e^{\mu_A(x_i)} + e^{\mu_A(x_i)}$$
$$+ (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}], \tag{3}$$

which vanishes at $\mu_A(x_i) = 0.5$.
Again differentiating (3) with respect to $\mu_A(x_i)$, we get

$$\frac{d^2H(A)}{d\mu_A(x_i)^2} = -\sum_{i=1}^{n} [2e^{\mu_A(x_i)} + \mu_A(x_i)e^{\mu_A(x_i)} + 2e^{(1-\mu_A(x_i))}$$
$$+ (1 - \mu_A(x_i))e^{(1-\mu_A(x_i))}],$$

and it is less than zero at $\mu_A(x_i) = 0.5$.
Hence H(A) is maximum at $\mu_A(x_i) = 0.5$ for all $i = 1, 2, ..., n$.

(P3) (Resolution): Let A^* be sharpened version of A, which means that
if $0 \leq \mu_A(x_i) < 0.5, \quad \mu_{A^*}(x_i) \leq \mu_A(x_i)$ for all $i = 1, 2, ..., n$ and
if $0.5 < \mu_A(x_i) \leq 1, \quad \mu_{A^*}(x_i) \geq \mu_A(x_i)$ for all $i = 1, 2, ..., n$.
Since $H(A)$ is an increasing function of $\mu_A(x_i)$ in the region $0 \leq \mu_A(x_i) < 0.5$ and $H(A)$ is a decreasing function of $\mu_A(x_i)$ in the region $0.5 < \mu_A(x_i) \leq 1$, therefore

(1) $\mu_{A^*}(x_i) \leq \mu_A(x_i) \Rightarrow H(A^*) \leq H(A)$ in $[0, \ 0.5]$
(2) $\mu_{A^*}(x_i) \geq \mu_A(x_i) \Rightarrow H(A^*) \leq H(A)$ in $[0.5, \ 1]$.

Hence above conditions together give

$$H(A^*) \leq H(A).$$

(P4) (Duality): It is evident from the definition that

$$H(\overline{A}) = H(A),$$

where $\overline{A}$ is complement of A obtained by replacing $\mu_A(x_i)$ by $1 - \mu_A(x_i)$.
Hence $H(A)$ satisfies all the essential four properties of fuzzy information measures. Thus it is valid exponential fuzzy information measure.
Proceeding on similar lines, it can easily be proved that the exponential measure $H_\alpha(A)$ proposed in (2) is a valid exponential fuzzy information measure of order α.

3. COMPARISONS AND RESULTS

The linguistic hedges, like "very", "more or less", "slightly", are habituated to represent the modifiers of linguistic variables. Fuzzy sets are conventionally utilized as linguistic variables. Thus, the hedges may be viewed as operations on fuzzy sets (visually perceive [26]). In this section, we consider these operations on fuzzy sets and then make comparisons of the proposed information measure of fuzzy sets with others.
For a given fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$, the modifier A^n for the fuzzy set A is defined as follows:

$$A^n = \{(x, (\mu_A(x))^n) \mid x \in X\}.$$

We then define the concentration and dilation of A with concentration: $CON(A) = A^2$ and dilation $DIL(A) = A^{1/2}$. The concentration and dilation are mathematical models frequently to be used for modifiers. Thus, we can use these mathematical operators to define the linguistic hedges on a fuzzy set A as follows

$$Very\, A = CON\,(A) = A^2, \ more \ or \ less\, A = DIL\,(A) = A^{1/2},$$

$$Quite \ Very\, A = A^3, \ Very \ Very\, A = A^4.$$

Let us consider a fuzzy set A_1 of $X = \{6, \ 7, \ 8, \ 9, \ 10\}$ is defined as

$$A_1 = \{(6, \ 0.1), (7, \ 0.3), (8, \ 0.4), (9, \ 0.9), (10, \ 1)\}.$$

By taking into account the characterization of linguistic variables, we regarded A as "Large" on X. We can generate the following fuzzy sets:

$$A_1^{1/2} = \{(6,\ 0.316), (7,\ 0.548), (8,\ 0.632), (9,\ 0.949), (10,\ 1)\},$$

$$A_1^2 = \{(6,\ 0.01),\ (7,\ 0.09),\ (8,\ 0.16),\ (9,\ 0.81),\ (10,\ 1)\},$$

$$A_1^3 = \{(6,\ 0.001), (7,\ 0.027), (8,\ 0.064), (9,\ 0.729), (10,\ 1)\},$$

$$A_1^4 = \{(6,\ 0),\ (7,\ 0.008),\ (8,\ 0.026),\ (9,\ 0.656),\ (10,\ 1)\}.$$

The hedges represented by the above fuzzy sets are described as follows:
$A_1^{1/2}$ may be treated as "More or Less Large",
A_1^2 may be treated as "Very Large",
A_1^3 may be treated as "Quite Very Large",
A_1^4 may be treated as "Very Very Large".
For the following information measures for fuzzy sets, the comparison of results are shown in Table 1:

$$H_K(A) = \frac{d_p(A,\ A_{near})}{d_p(A,\ A_{far})};\ [9]$$

$$H_{Pal}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)e^{1-\mu_A(x_i)}\ + (1-\mu_A(x_i))\,e^{\mu_A(x_i)}\right];\ [17]$$

$$H_{LL}(A) = \sum_{i=1}^{n} S\left(C_r(\xi_A = x_i)\right);\ [12]$$

$$H_{HY}(A) = \frac{1}{(1-e^{-1/2})}\sum_{i=1}^{n}\left[\left(1-e^{-\mu_{A^c}(x_i)}\right) I_{[\mu_A(x_i)\geq 1/2]}\right.$$

$$\left. + \left(1-e^{-\mu_A(x_i)}\right) I_{[\mu_A(x_i)<1/2]}\right];\quad [7]$$

$$H(A) = \sum_{i=1}^{n}\left[e - \mu_A(x_i)e^{\mu_A(x_i)} - (1-\mu_A(x_i))e^{(1-\mu_A(x_i))}\right].$$

Table 1 Comparison of the measures of fuzziness with different information measures

Fuzzy Set	$H_K(A)$	$H_{Pal}(A)$	$H_{LL}(A)$	$H_{HY}(A)$	$H(A)$
$A_1^{1/2}$	0.220	1.389	0.810	0.505	3.200
A_1	0.311	1.331	0.723	0.397	2.720
A_1^2	0.099	1.202	0.378	0.212	1.655
A_1^3	0.078	1.151	0.870	0.167	1.236
A_1^4	0.082	1.1.36	0.692	0.165	1.116

From the Table 1, we see that

$$H_K(A_1^{1/2}) < H_K(A_1) > H_K(A_1^2) > H_K(A_1^3) < H_K(A_1^4),$$

$$H_{Pal}(A_1^{1/2}) > H_{Pal}(A_1) > H_{Pal}(A_1^2) > H_{Pal}(A_1^3) > H_{Pal}(A_1^4),$$

$$H_{LL}(A_1^{1/2}) > H_{LL}(A_1) > H_{LL}(A_1^2) < H_{LL}(A_1^3) > H_{LL}(A_1^4),$$

$$H_{HY}(A_1^{1/2}) > H_{HY}(A_1) > H_{HY}(A_1^2) > H_{HY}(A_1^3) > H_{HY}(A_1^4),$$

$$H(A_1^{1/2}) > H(A_1) > H(A_1^2) > H(A_1^3) > H(A_1^4).$$

Consequently, on the basis of above results of the information measures $H_{Pal}(A)$, $H_{HY}(A)$ and $H(A)$ represents better ramifications than others. Further, we display the result only for $H_{Pal}(A)$, $H_{HY}(A)$ and $H(A)$.
Let us consider another fuzzy set A_2 of $X = \{6,\ 7,\ 8,\ 9,\ 10\}$ defined by

$$A_2 = \{(6,\ 0.2),\ (7,\ 0.3),\ (8, 0.4),\ (9, 0.7),\ (10,\ 0.8)\}.$$

Table 2 Comparison of the measures of fuzziness $H_{Pal}(A)$, $H_{HY}(A)$ and $H(A)$

Fuzzy Set	$A_2^{1/2}$	A_2	A_2^2	A_2^3	A_2^4
$H_{Pal}(A)$	1.501	1.513	1.386	1.094	1.241
$H_{HY}(A)$	0.653	0616	0.577	0.393	0.298
$H(A)$	4.299	4.223	3.171	2.451	1.971

We generate $A_2^{1/2}$, A_2^2, A_2^3, A_2^4 using above mentioned notion. The results for the information measures $H_{Pal}(A)$, $H_{HY}(A)$ and $H(A)$ are exhibit in Table 2. From the Table 2, the consequences of ramifications are

$$H_{Pal}(A_2^{1/2}) > H_{Pal}(A_2) > H_{Pal}(A_2^2) > H_{Pal}(A_2^3) > H_{Pal}(A_2^4),$$

$$H_{HY}(A_2^{1/2}) > H_{HY}(A_2) > H_{HY}(A_2^2) > H_{HY}(A_2^3) > H_{HY}(A_2^4),$$

$$H(A_2^{1/2}) > H(A_2) > H(A_2^2) > H(A_2^3) > H(A_2^4).$$

Consequently, the order of the proposed information measure $H(A)$ and $H_{HY}(A)$ represent better ramifications than $H_{Pal}(A)$. From the anterior comparisons, the proposed information measure is authentically better for representing the quantification of fuzziness for fuzzy sets.

4. EXPONENTIAL FUZZY DISCRIMINATION MEASURES

[11] defined the measure of directed divergence of probability distribution P from the probability distribution Q as

$$D(P:Q) = \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}. \tag{4}$$

[10] also suggested the measure of symmetric divergence as

$$J(P:Q) = \sum_{i=1}^{n}(p_i - q_i)\log\frac{p_i}{q_i}. \tag{5}$$

The simplest fuzzy directed divergence measure and symmetric divergence measure suggested by [1] are

$$I_{BP}(A,B) = \sum_{i=1}^{n}[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+(1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}], \tag{6}$$

and

$$D_{BP}(A,B) = \sum_{i=1}^{n}\left[\{\mu_A(x_i) - \mu_B(x_i)\}\log\frac{\mu_A(x_i)(1-\mu_B(x_i))}{\mu_B(x_i)(1-\mu_A(x_i))}\right]. \tag{7}$$

Thus (7) can be written as

$$D_{BP}(A,B) = I_{BP}(A,B) + I_{BP}(B,A) \qquad (8)$$

Let us consider $A,\ B \in FS(X)$, then the exponential measures of information discrimination of A against B based solely on the membership functions $\mu_A(x_i)$ and $\mu_B(x_i)$ are given by

$$I^{'}(A,B) = \sum_{i=1}^{n}\left[\mu_A(x_i)e^{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + (1-\mu_A(x_i))e^{\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}} - e\right] \qquad (9)$$

and

$$I^{'}_{\alpha}(A,B) = \sum_{i=1}^{n}[\mu_A(x_i)e^{\alpha\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)}$$

$$+(1-\mu_A(x_i))e^{\alpha\left(\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right)} - e^{\alpha}], \qquad (10)$$

where $\alpha > 0, \alpha \neq 1$.

However, it may be noted that (9) and (10) are undefined if $\mu_B(x_i) = 0$ for any $x_i \in X$.

So, in view of [13], (9) and (10) are taken as given below:

$$I(A,B) = \sum_{i=1}^{n}\mu_A(x_i)e^{\frac{\mu_A(x_i)}{1/2(\mu_A(x_i)+\mu_B(x_i))}}$$

$$+(1-\mu_A(x_i))e^{\frac{(1-\mu_A(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_B(x_i))\right\}}} - e], \qquad (11)$$

and

$$I_{\alpha}(A,B) = \sum_{i=1}^{n}[\mu_A(x_i)e^{\alpha\left(\frac{\mu_A(x_i)}{1/2\{\mu_A(x_i)+\mu_B(x_i)\}}\right)}$$

$$+(1-\mu_A(x_i))e^{\alpha\left(\frac{(1-\mu_A(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_B(x_i))\right\}}\right)} - e^{\alpha}]. \qquad (12)$$

Since $I(A,B)$ and $I_{\alpha}(A,B)$ are not symmetric with respect to its arguments, therefore, the corresponding exponential fuzzy symmetric discrimination measures are defined as

$$D(A,B) = I(A,B) + I(B,A) \qquad (13)$$

and

$$D_{\alpha}(A,B) = I_{\alpha}(A,B) + I_{\alpha}(B,A). \qquad (14)$$

To prove the measures $I(A,B)$ and $I_{\alpha}(A,B)$ are valid measures, we show that $I(A,B) \geq 0$ and $I_{\alpha}(A,B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1, 2, ..., n$.

Let $\sum_{i=1}^{n}\mu_A(x_i) = s$ and $\sum_{i=1}^{n}\mu_B(x_i) = t$, then

$$\sum_{i=1}^{n}[\mu_A(x_i)e^{\frac{\mu_A(x_i)}{1/2(\mu_A(x_i)+\mu_B(x_i))}}$$

$$+(1-\mu_A(x_i))e^{\frac{(1-\mu_A(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_B(x_i))\right\}}} - e]$$

$$\geq \left[s\,e^{\frac{s}{1/2(s+t)}} + (n-s)e^{\frac{(n-s)}{\left\{n-1/2(s+t)\right\}}} - n\,e\right]. \qquad (15)$$

From (15), we have

$$I(A,B) \geq \left[s\,e^{\frac{2s}{(s+t)}} + (n-s)e^{\frac{2(n-s)}{(2n-(s+t))}} - n\,e\right].$$

Further Let $\psi(s) = \left[s\,e^{\frac{2s}{(s+t)}} + (n-s)e^{\frac{2(n-s)}{(2n-(s+t))}} - n\,e\right]$, then

$$\psi^{'}(s) = \left[e^{\frac{2s}{(s+t)}} + \frac{2st}{(s+t)^2}e^{\frac{2s}{(s+t)}} - e^{\frac{2(n-s)}{\{2n-(s+t)\}}}\right.$$

$$\left. - \frac{2(n-s)(n-t)}{\{2n-(s+t)\}^2}e^{\frac{2(n-s)}{\{2n-(s+t)\}}}\right],$$

and

$$\psi^{''}(s) = \left\{e^{\frac{2s}{(s+t)}}\left(\frac{2t}{(s+t)^2} + \frac{2t(t-s)}{(s+t)^3} + \frac{4st^2}{(s+t)^4}\right)\right.$$

$$\left. +e^{\frac{2(n-s)}{\{2n-(s+t)\}}}\left(\frac{2(n-t)}{\{2n-(s+t)\}^2} + \frac{2(n-s)(n-t)}{\{2n-(s+t)\}^3} + \frac{4(n-s)(n-t)^2}{\{2n-(s+t)\}^4}\right)\right.$$

which shows that $\psi(s)$ is convex function of s whose minimum values arise when $\frac{s}{t} = \frac{n-s}{n-t} = 1$ and are equal to zero. Hence $\psi(s) > 0$ vanishes only when $s = t$.

Hence $I(A,B) > 0$ and it vanishes when $A = B$. Thus (11) is valid discrimination information measure for $FS(X)$. Consequently, $D(A,B)$ is valid symmetric discrimination information measure for $FS(X)$.

Similarly, it can easily be proved that the exponential fuzzy discrimination measure $I_{\alpha}(A,B)$ proposed in (12) is a valid fuzzy discrimination measure of order α. Consequently, $D_{\alpha}(A,B)$ is valid symmetric discrimination measure for $FS(X)$ of order α.

Now, we study the important properties of the exponential fuzzy discrimination measures $I(A,B)$ and $I_{\alpha}(A,B)$ given by (10) and (11) respectively.

Theorem 2. Let A and B be two fuzzy sets in a fixed universe of discourse X. Let $A(x_i) = \mu_A(x_i)$ and $B(x_i) = \mu_B(x_i)$ satisfying either $A \subseteq B$ or $B \subseteq A$, then the following holds:

(a). $I(A \cup B, A) + I(A \cap B, A) = I(B, A)$;

(b). $I(A \cup B, C) + I(A \cap B, C) = I(A, C) + I(B, C)$;

(c). $I((A \cup B)^c, (A \cap B)^c) = I(A^c \cup B^c, A^c \cap B^c)$.

Proof: Let us separate X into X_1 and X_2, where

$X_1 = \{x \in X : A(x) \subseteq B(x)\}$ and

$X_2 = \{x \in X : B(x) \subseteq A(x)\}$. It implies that

$X_1 = \{x_i : \mu_A(x_i) \geq \mu_B(x_i), \forall\, x_i \in X\}$ and

$X_2 = \{x_i : \mu_A(x_i) < \mu_B(x_i), \forall\, x_i \in X\}$.

Using the notions elaborated above in introduction, we have

In set X_1,

$A \cup B =$ Union of A and B

$\Leftrightarrow \mu_{A\cup B}(x_i) = \max\{\mu_A(x_i),\ \mu_B(x_i)\} = \mu_A(x_i)$,

$A \cap B =$ Intersection of A and B

$\Leftrightarrow \mu_{A\cap B}(x_i) = \min\{\mu_A(x_i), \mu_B(x_i)\} = \mu_B(x_i)$.

In set X_2,

$A \cup B =$ Union of A and B

$\Leftrightarrow \mu_{A\cup B}(x_i) = \max\{\mu_A(x_i), \mu_B(x_i)\} = \mu_B(x_i)$,

$A \cap B =$ Intersection of A and B

$\Leftrightarrow \mu_{A\cap B}(x_i) = \min\{\mu_A(x_i), \mu_B(x_i)\} = \mu_A(x_i)$.
(a).

$$I(A\cup B, A) = \sum_{i=1}^{n}[\mu_{A\cup B}(x_i)e^{\frac{\mu_{A\cup B}(x_i)}{1/2(\mu_{A\cup B}(x_i)+\mu_B(x_i))}}$$

$$+(1-\mu_{A\cup B}(x_i))\,e^{\frac{(1-\mu_{A\cup B}(x_i))}{\left\{1-1/2(\mu_{A\cup B}(x_i)+\mu_B(x_i))\right\}}} - e],$$

$$= \sum_{x_i\in X_1}[\mu_A(x_i)e^{\frac{\mu_A(x_i)}{1/2(\mu_A(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_A(x_i))\,e^{\frac{(1-\mu_A(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_A(x_i))\right\}}} - e]$$

$$+\sum_{x_i\in X_2}[\mu_B(x_i)e^{\frac{\mu_B(x_i)}{1/2(\mu_B(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_B(x_i))\,e^{\frac{(1-\mu_B(x_i))}{\left\{1-1/2(\mu_B(x_i)+\mu_A(x_i))\right\}}} - e]. \qquad (16)$$

Similarly,

$$I(A\cap B, A) = \sum_{i=1}^{n}[\mu_{A\cap B}(x_i)e^{\frac{\mu_{A\cap B}(x_i)}{1/2(\mu_{A\cap B}(x_i)+\mu_B(x_i))}}$$

$$+(1-\mu_{A\cap B}(x_i))\,e^{\frac{(1-\mu_{A\cap B}(x_i))}{\left\{1-1/2(\mu_{A\cap B}(x_i)+\mu_B(x_i))\right\}}} - e],$$

$$= \sum_{x_i\in X_1}[\mu_B(x_i)e^{\frac{\mu_B(x_i)}{1/2(\mu_B(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_B(x_i))\,e^{\frac{(1-\mu_B(x_i))}{\left\{1-1/2(\mu_B(x_i)+\mu_A(x_i))\right\}}} - e]$$

$$+\sum_{x_i\in X_2}[\mu_A(x_i)e^{\frac{\mu_A(x_i)}{1/2\,(\mu_A(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_A(x_i))\,e^{\frac{(1-\mu_A(x_i))}{\left\{1-1/2\,(\mu_A(x_i)+\mu_A(x_i))\right\}}} - e]. \qquad (17)$$

From (16) and (17), we have

$$I(A\cup B, A) + I(A\cap B, A)$$

$$= \sum_{x_i\in X_1}[\mu_B(x_i)e^{\frac{\mu_B(x_i)}{1/2(\mu_B(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_B(x_i))\,e^{\frac{(1-\mu_B(x_i))}{\left\{1-1/2(\mu_B(x_i)+\mu_A(x_i))\right\}}} - e]$$

$$+\sum_{x_i\in X_2}[\mu_B(x_i)e^{\frac{\mu_B(x_i)}{1/2(\mu_B(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_B(x_i))\,e^{\frac{(1-\mu_B(x_i))}{\left\{1-1/2(\mu_B(x_i)+\mu_A(x_i))\right\}}} - e],$$

$$= \sum_{i=1}^{n}[\mu_B(x_i)e^{\frac{\mu_B(x_i)}{1/2(\mu_B(x_i)+\mu_A(x_i))}}$$

$$+(1-\mu_B(x_i))\,e^{\frac{(1-\mu_B(x_i))}{\left\{1-1/2(\mu_B(x_i)+\mu_A(x_i))\right\}}} - e],$$

$= I(B, A)$.
Thus

$$I(A\cup B, A) + I(A\cap B, A) = I(B, A).$$

Similarly, (b) and (c) can be proved.

Theorem 3. Let $A, B \in FS(X)$, and $A(x_i) = \mu_A(x_i)$ and $B(x_i) = \mu_B(x_i)$ satisfying either $A \subseteq B$ or $B \subseteq A$, and $\alpha > 0$ and $\alpha \neq 1$, then the following holds:
(a). $I_\alpha(A\cup B, A) + I_\alpha(A\cap B, A) = I_\alpha(B, A)$;
(b). $I_\alpha(A\cup B, C) + I_\alpha(A\cap B, C) = I_\alpha(A, C) + I_\alpha(B, C)$;
(c). $I_\alpha((A\cup B)^c, (A\cap B)^c) = I_\alpha(A^c\cup B^c, A^c\cap B^c)$.
Proof: The results (a) to (c) are evident and can easily be verified.

Theorem 4. Let A and B be two fuzzy sets in a fixed universe of discourse$X$, then it satisfies the following
(a). $I(A, A^c) = I(A^c, A)$;
(b). $I(A^c, B^c) = I(A, B)$;
(c). $I(A, B^c) = I(A^c, B)$
(d). $I(A, B) + I(A^c, B) = I(A^c, B^c) + I(A, B^c)$.
Proof:
(a).

$$I(A, A^c) = \sum_{i=1}^{n}[\mu_A(x_i)e^{\frac{\mu_A(x_i)}{1/2(\mu_A(x_i)+\mu_{A^c}(x_i))}}$$

$$+(1-\mu_A(x_i))\,e^{\frac{(1-\mu_A(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_{A^c}(x_i))\right\}}} - e],$$

$$I(A, A^c) = \sum_{i=1}^{n}\left[\mu_A(x_i)e^{2\mu_A(x_i)} + (1-\mu_A(x_i))\,e^{2(1-\mu_A(x_i))} - e\right] \qquad (18)$$

and

$$I(A^c, A) = \sum_{i=1}^{n}[\mu_{A^c}(x_i)e^{\frac{\mu_{A^c}(x_i)}{1/2(\mu_A(x_i)+\mu_{A^c}(x_i))}}$$

$$+(1-\mu_{A^c}(x_i))\,e^{\frac{(1-\mu_{A^c}(x_i))}{\left\{1-1/2(\mu_A(x_i)+\mu_{A^c}(x_i))\right\}}} - e]$$

$$I(A,A^c)=\sum_{i=1}^{n}\left[\mu_A(x_i)e^{2\mu_A(x_i)}+(1-\mu_A(x_i))\,e^{2(1-\mu_A(x_i))}-e\right]. \tag{19}$$

From (18) and (19), we get

$$I(A,A^c)=I(A^c,A).$$

Proceeding on the similar line, we can prove (b), (c) and (d).

Theorem 5. Let $A,\ B \in FS(X)$, $\alpha > 0$ and $\alpha \neq 1$, then the following holds:
(a). $I_\alpha\left(A,A^c\right)=I_\alpha\left(A^c,A\right)$;
(b). $I_\alpha\left(A^c,B^c\right)=I_\alpha\left(A,B\right)$;
(c). $I_\alpha\left(A,B^c\right)=I_\alpha\left(A^c,B\right)$;
(d). $I_\alpha\left(A,B\right)+I_\alpha\left(A^c,B\right)=I_\alpha\left(A^c,B^c\right)+I_\alpha\left(A,B^c\right)$.
Proof: The results (a) to (d) are evident and can effortlessly be verified.

5. APPLICATIONS

Fuzzy Sets are suitable tools to cope with imperfectly defined facts and data as well as with imprecise knowledge. In this section, we discuss the application of the exponential fuzzy symmetric discrimination measure given by (13) to pattern recognition and diagnosis of crop diseases.

5.1 Pattern Recognition

Let us consider the problem of three known patterns P_1, P_2 and P_3, which have classifications C_1, C_2 and C_3 respectively.
The patterns are represented by the following FSs in $X = \{x_1,x_2,x_3\}$:

$$P_1=\{(x_1,0.9),(x_2,0.8),(x_3,0.7)\}, \tag{20}$$

$$P_2=\{(x_1,0.8),(x_2,0.9),(x_3,0.9)\}, \tag{21}$$

$$P_3=\{(x_1,0.6),(x_2,0.8),(x_3,0.9)\}. \tag{22}$$

Given an unknown pattern Q, represented by the FS

$$Q=\{(x_1,0.5),(x_2,0.6),(x_3,0.8)\}. \tag{23}$$

Here our aim is to classify Q to one of the classes C_1, C_2 and C_3. According to the principle of minimum discrimination measure between FSs, the process of assigning Q to C_{k^*} is described by

$$k^*=\arg\min_k\{D\left(P_k,Q\right)\}. \tag{24}$$

Table 3 presents $D(P_k,Q),\ k\in\{1,2,3\}$. We can observe that Q has correctly being classified to C_3.

Table 3 Symmetric exponential fuzzy discrimination measure $D\left(P_k,Q\right)$, with $k\in\{1,2,3\}$

	P_1	P_2	P_3
Q	2.1292	1.3127	**0.6374**

Thus, Q has classification C_3, since $k^*=0.6374$ is minimum.

5.2 Diagnosis of Crop Diseases

The exponential fuzzy symmetric discrimination measure is applied to diagnose the common diseases in the crops. For that let us consider the data of agriculture department provided by Belahata gram panchayat of Satna District (M. P.), India.
The data consists of the set of crops C = {Wheat, Rice, Onion red, Carrot}, the set of diseases D = {Bacterial, Fungal, Nematodes, Viroid, Phytoplasmal} and the set of factors F = {Temperature, Soil moisture, Insect, pH value, Humidity} mainly affecting the crops in the region. Table 4 presents the associated characteristic for the various diseases. The factors required for each crops are given in Table 5. Each element of the table is given in the form of the membership values, e.g., in Table 4 the temperature for bacterial is described by $(\mu=0.6)$.

Table 4 Set of Symptoms Characteristic for Diagnosis

	Bactarial	Fungal	Nematodes	Viroid	Phytoplasmal
Temperature	0.6	0.8	0.5	0.4	0.7
Soil Moisture	0.7	0.6	0.9	0.6	0.2
Insect	0.8	0.6	0.3	0.6	0.9
pH value	0.8	0.5	0.5	0.6	1.0
Humidity	0.8	0.9	0.7	1.0	0.3

Table 5 Set of Symptoms Characteristic for the Crop Considered

	Temperature	Soil moisture	Insect	pH value	Humidity
Wheat	0.6	0.6	0.8	0.7	0.6
Rice	0.8	1.0	0.9	0.9	0.9
Onion red	0.9	0.8	0.5	0.5	0.8
Carrot	0.4	0.9	0.6	0.6	0.9

For proper diagnosis, we calculate $c_i \in C$ for each crop, where $i\in\{1,2,3,4\}$, the exponential fuzzy symmetric discrimination measure for FSs $D\left(f(c_i),d_k\right)$ between crop symptoms and the set of symptoms that are characteristic for each diagnosis $d_k \in D$ with $k \in \{1,......,5\}$. Similarly to (24), the proper diagnosis d_{k^*} for the ith crop is derived as given below:

$$k^*=\arg\min_k\{D\left(f(c_i),d_k\right)\} \tag{25}$$

We assign to the ith crop the diagnosis whose symptoms have the lowest exponential fuzzy symmetric discrimination measure from crop symptoms. The results for the crops considered are given in Table 6.

Table 6 Symmetric exponential Fuzzy Discrimination Measure $D_j\left(f(c_i),d_k\right)$ among each Crop's Symptoms and the Considered Set of Possible Diagnoses

	Bacterial	Fungal	Nematodes	Viroid	Phytoplasmal
Wheat	**0.3171**	1.1558	1.8697	1.8950	2.3660
Rice	**0.9180**	1.8679	2.8611	2.5981	4.4129
Onion red	1.3647	**0.3835**	0.9964	1.9773	5.6814
Carrot	0.8800	1.2097	**0.6862**	0.7607	5.9645

From the Table 6, we infer that Wheat and Rice are most affected by Bacterial Disease, Onion red is most affected by Fungal and Carrot is affected by Nematodes.

6. CONCLUSIONS AND FUTURE WORK

In this paper, two new exponential fuzzy information measures are introduced and proved their validity. The exponential fuzzy discrimination measures and exponential fuzzy symmetric discrimination measures of these exponential fuzzy information measures are also developed. Further, the properties of the exponential fuzzy symmetric discrimination measures are listed and finally, the proposed exponential fuzzy symmetric discrimination measure has been applied to problem of pattern recognition. Diagnosis of crop disease using exponential fuzzy symmetric discrimination measure is done. The method provides the maximum possibility of the crop to be affected by particular diseases. It is worth mentioning that exponential fuzzy symmetric discrimination measures given by (13) and (14) have lot of potentiality for application in study of various new challenges of pattern recognition, medical diagnosis, image processing etc.

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