Weak Set-Labeling Number of Certain Integer Additive Set-Labeled Graphs

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ABSTRACT

Let $\mathbb{N}_0$ be the set of all non-negative integers, let $X \subset \mathbb{N}_0$ and $\mathcal{P}(X)$ be the the power set of X. An integer additive set-labeling (IASL) of a graph G is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sum set of $f(u)$ and $f(v)$. An IASL f is said to be an integer additive set-indexer (IASI) of a graph G if the induced edge function f^+ is also injective. An integer additive set-labeling f is said to be a weak integer additive set-labeling (WIASL) if $|f^+(uv)| = \max(|f(u)|, |f(v)|)\ \forall\ uv \in E(G)$. The minimum cardinality of the ground set X required for a given graph G to admit an IASL is called the set-labeling number of the graph. In this paper, the notion of the weak set-labeling number of a graph G is introduced as the minimum cardinality of X so that G admits a WIASL with respect to the ground set X and the weak set-labeling numbers of certain graphs are discussed.

Keywords:

Integer additive set-labeled graphs; weak integer additive set-labeled graphs; weak set-labeling number of a graph.

AMS Subject Classification: 05C78

1. INTRODUCTION

For all terms and definitions, not defined specifically in this paper, refer to [1], [6] and [14] and for different graph classes, further refer to [2] and [3]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The sum set of two sets A and B, denoted $A + B$, is the set defined by $A + B = \{a + b : a \in A, b \in B\}$. If either A or B is countably infinite, then their sum set will also be countably infinite. Hence, all sets that are considered in this study are finite sets. The cardinality of a set A is denoted by $|A|$. The power set of a set A is denoted by $\mathcal{P}(A)$.

Using the concepts of sumsets, the notion of an integer additive set-labeling of a graph G is introduced as follows.

Let $\mathbb{N}_0$ denote the set of all non-negative integers and X be a subset of $\mathbb{N}_0$. An *integer additive set-labeling* (IASL, in short) of a graph G is defined as an injective function $f : V(G) \to \mathcal{P}(X)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(X)$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumset of the set-labels $f(u)$ and $f(v)$. A graph which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

An *integer additive set-indexer* of a graph G is an integer additive set-labeling $f : V(G) \to \mathcal{P}(X)$ such that the induced edge function $f^+ : E(G) \to \mathcal{P}(X)$ is also injective.

Figure 1 depicts an integer additive set-labeling defined on a given graph G.

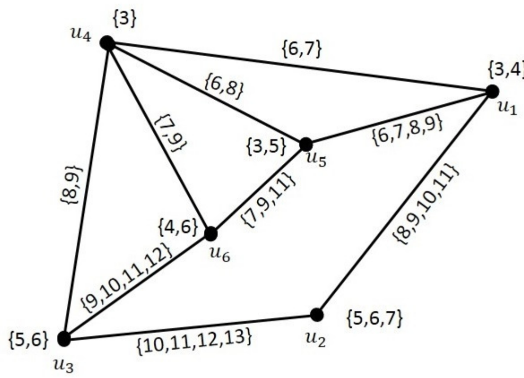

Fig. 1. An illustration to an IASL-graph.

The *set-labeling number* of a given graph G is the minimum cardinality of the ground set X so that the function $f : V(G) \to \mathcal{P}(X)$ is a WIASL of G. The set-labeling number of a graph G is denoted by $\sigma(G)$.

DEFINITION 1.1. [8] A *weak integer additive set-labeling* f of a graph G is an IASL such that $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A weak IASI f is said to be *weakly uniform IASI* if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer k. A graph which admits a weak IASI may be called a *weak integer additive set-labeled graph* (WIASL-graph).

A WIASL f of a given graph G is said to be a *weakly k-uniform IASL* (k-uniform WIASL) of G if the set-labels of all edges of G

have the same cardinality k, where k is a positive integer. If G admits a WIASL, then it can be noted that the vertex set of G can be partitioned into two sets such that the first set, say V_1, consists of all those vertices of G having singleton set-labels and the other set, say V_2 consists of all those vertices having non-singleton set-labels. As a result, the following theorem have been established.

THEOREM 1.2. [5] *A graph G admits a k-uniform WIASL if and only if G is bipartite or $k = 1$.*

The following result is a necessary and sufficient condition for a given graph to admit a weak integer additive set-labeling.

LEMMA 1.3. [8] *A graph G admits a weak integer additive set-indexer if and only if every edge of G has at least one mono-indexed end vertex.*

In view of the above lemma, no two of its adjacent vertices can have non-singleton set-labels. Then, some of the adjacent vertices of G can have singleton sets as their end vertices. In these cases, the set-label of the corresponding edges are also singleton sets. As a result, one can have the following theorem.

THEOREM 1.4. [8] *A graph G admits a WIASL if and only if G is bipartite or it has some edges having singleton set-label.*

In view of the above results, one can re-define a WIASL f of a given graph G as an IASL with respect to which the cardinality of the set-label of every edge of G is equal to the cardinality of the set-label of at least one of its end vertices.

The following result is another important observation on WIASL-graphs.

THEOREM 1.5. [10] *Let G be a WIASL-graph. Then, the minimum number of vertices of G having singleton set-labels is equal to the vertex covering number α of G and the maximum number of vertices of G having non-singleton set-labels is equal to the independence number β of G.*

Figure 2 illustrates a WIASL defined on a given graph G.

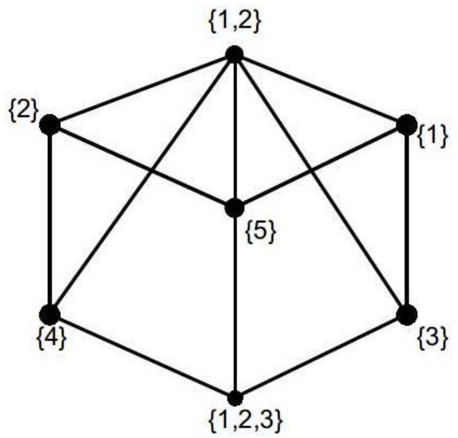

Fig. 2. An example to a WIASL-Graph

2. WEAK SET-LABELING NUMBER OF GRAPHS

Analogous to the terminology of set-labeling number of graphs which admit integer additive set-labelings, the notion of weak set-labeling number of a given graph G is introduced as follows.

DEFINITION 2.1. Let a function $f : V(G) \to \mathcal{P}(X)$ be an IASL of a given graph G, where X is a non-empty finite ground set of non-negative integers. Then, the *weak set-labeling number* of a graph G is the minimum cardinality of the ground set X, such that f is a WIASL of G. The weak set-labeling number of a graph G is denoted by $\sigma^*(G)$.

No IASL-graphs considered in the following discussion are 1-uniform, unless specified otherwise.
In the following discussion, the weak set-labeling numbers of different standard graphs are determined. Let us begin with a path graph P_n on n vertices.

THEOREM 2.2. *The weak set-labeling number of a path P_n is* $2 + \lfloor \frac{n}{2} \rfloor$.

PROOF. Let $\{v_i, v_2, v_3, \ldots, v_n\}$ be the vertex set of the graph P_n. Note that a set-labeling in such a way that no two adjacent vertices in P_n can have non-singleton set-labels is required. Hence, the vertices of P_n are to labeled alternately by singleton and non-singleton sets of non-negative integers. In this context, the following cases are to be considered.

Case-1: Let n be even. Then, $n = 2r$ for some positive integer r. Label the vertices of P_n by singleton sets and 2-element sets of non-negative integers as follows. Let $f(v_2) = \{1\}, f(v_4) = \{2\}, f(v_6) = \{3\}, \ldots, f(v_{2r}) = \{r\}$. Now, label odd vertices in reverse order as follows. Let $f(v_{2r-1}) = \{1,2\}, f(v_{2r-3}) = \{2,3\}, f(v_{2r-5}) = \{3,4\}, \ldots, v_3 = \{r-1,r\}, f(v_1) = \{r,1\}$. Then, the set-labels of the edges of G are $v_nv_{n-1} = \{r+1, r+2\}$ $v_{n-1}v_{n-2} = \{1+r-1, 2+r-1\} = \{r+1, r+2\}$ and so on. The set-labels of all edges other than the edge v_1v_2 are either {r+1,r+2}$or\{r, r+1\}$ and the set-label of v_1v_2 is $\{2, r+1\}$. That is, $f^+(E) = \{\{2, r+1\}, \{r, r+1\}, \{r+1, r+2\}\}$. Now, choose $X = f(V) \cup f^+(E) = \{1, 2, 3, \ldots, r+1, r+2\}$. Clearly, $f : V(G) \to \mathcal{P}(X)$ is a WIASL defined on P_n and hence, $\sigma^*(P_n) = 2 + \frac{n}{2}$.

Case-2: Let n be odd. Then, $n = 2r+1$, for some positive integer r. Label the vertices of P_n as follows. Let $f(v_2) = \{1\}, f(v_4) = \{2\}, f(v_6) = \{3\}, \ldots, f(v_{2r}) = \{r\}$. Now, label the remaining vertices in the reverse order as follows. Let $f(v_{2r+1}) = \{1,2\}, f(v_{2r-1}) = \{2,3\}, f(v_{2r-3}) = \{3,4\}, \ldots, v_5 = \{r-1, r\}, f(v_3) = \{r,1\}$ and $f(v_1) = \{1,2,3\}$. Therefore, as in the previous case, $f^+(E) = \{\{2,3,4\}, \{r, r+1\}, \{r+1, r+2\}\}$. Then, $X = f(V) \cup f^+(E) = \{1, 2, 3, \ldots, r+1, r+2\}$. Therefore, the weak set-labeling number of P_n is $r + 2 = 2 + \frac{n-1}{2}$.

Combining the above two cases, $\sigma^*(P_n) = 2 + \lfloor \frac{n}{2} \rfloor$. □

Next, the weak set-labeling number of cycle graphs is determined in the following way.

THEOREM 2.3. *The weak set-labeling number of a cycle C_n is* $2 + \lfloor \frac{n}{2} \rfloor$.

PROOF. Let $C_n : v_iv_2v_3 \ldots v_nv_1$ be a cycle on n vertices. We label the vertices alternately by singleton and non-singleton sets of non-negative integers. Here, the following cases are to be considered.

Case-1: Let n be even. Then, $n = 2r$ for some positive integer r. Label the vertices of C_n by singleton sets and 2-element sets of non-negative integers as follows. Let $f(v_{2i}) = \{i\}$ for all $1 \leq i \leq \frac{n}{2}$. Label the odd vertices such that $f(v_{2r-1}) = \{1,2\}, f(v_{2r-3}) = \{2,3\}, f(v_{2r-5}) = \{3,4\}, \ldots, v_3 = \{r-1,r\}, f(v_1) = \{r,1\}$. Then, all edges of C_n except v_1v_2 and v_2v_3 have the set-labels either $\{r,r+1\}$ or $\{r+1,r+2\}$. Also, $f(v_1v_2) = \{2,r+1\}$ and $f(v_2v_3) = \{3,r+2\}$. Therefore, Choose $X = \{1,2,3,\ldots,r+1,r+2\}$. Hence, in this case $\sigma^*(C_n) = r+2 = 2+\frac{n}{2}$.

Case-2: Let n be odd. Then, $n = 2r+1$, for some positive integer r. For an WIASL-graph C_n, there exist $r+1$ singleton set labels and r non-singleton sets. Label the vertices of C_n as follows. Now, let $f(v_1) = \{1\}, f(v_3) = \{2\}, f(v_5) = \{3\}, \ldots, f(v_{2r+1}) = \{r+1\}$ and let $f(v_2) = \{r,r-1\}, f(v_4) = \{r-1,r-2\}, f(v_6) = \{r-2,r-3\}, \ldots f(v_{2r}) = \{1,2\}$. Then, all edges other than the edge v_nv_1 have the set-label either $\{r,r+1\}$ or $\{r+1,r+2\}$ and the edge v_nv_1 has the set-label $\{r+2\}$. Therefore, the set X can be chosen as the set $f(V) \cup f^+(E) = \{1,2,3,\ldots,r+1,r+2\}$ and hence $\sigma^*(G) = r+2 = 2+\frac{n-1}{2}$.

Combining the above two cases, one can note that $\sigma^*(C_n) = 2 + \lfloor \frac{n}{2} \rfloor$. □

The weak set-labeling number of a complete graph is determined in the following theorem.

THEOREM 2.4. *The weak set-labeling number of a complete graph K_n is $2n-3$.*

PROOF. . Let $V(K_n) = \{v_1,v_2,v_3,\ldots,v_n\}$. Since all vertex of K_n are mutually adjacent, at most one vertex can have a non-singleton set-label. Let f be an IASL defined on K_n which assigns the set-labels to the vertices of K_n as follows. Let $f(v_i) = \{i\}$ for $1 \leq i \leq n-1$ and let $f(v_n) = \{1,2\}$. Then, $f^+(E(K_n)) = \{3,4,5,\ldots,2n-3\}$. That is, the ground set with minimum cardinality is the set $f(V) \cup f^+(E) = \{1,2,3,\ldots,2n-3\}$. Hence, $\sigma^*(G) = 2n-3$. □

Figure 3 depicts a complete graph on 6 vertices and weak set-labeling number 9.

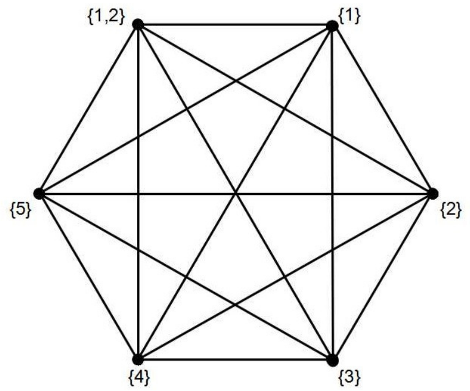

Fig. 3. A complete graph with weak set-labeling number 9.

3. WEAK SET-LABELING NUMBER OF CERTAIN GRAPH CLASSES

Let us now proceed to discuss the weak set-labeling number of certain graphs that are generated from cycles. A graph that needs to be considered first among these types of graphs is a *wheel graph* W_{n+1} which is a graph obtained by drawing edges from all vertices of a cycle to an external vertex (see [3]). That is, $W_{n+1} = C_n + K_1$. The following theorem establishes the weak set-labeling number of a wheel graph.

THEOREM 3.1. *The weak set-labeling number of a wheel graph $W_{n+1} = C_n + K_1$ is $3 + \lfloor \frac{n}{2} \rfloor$.*

PROOF. Let $G = C_n + K_1$, where $C_n : v_1v_2v_3\ldots v_nv_1$ and $K_1 = \{v\}$. Label the vertices of C_n in G as explained in Theorem 2.3. What remains is to label the vertex v. Since all sets of the form $\{i,i+1\}$ have already been used for labeling the vertices of C_n, label v by the set $\{1,3\}$. It can be observed that the only element in a set-label of the edge vv_i of G, which is not in any set-label of the elements of C_n is $r+3$, where $r = \lfloor \frac{n}{2} \rfloor$. Hence, $f(V(G)) \cup f^+(E(G)) = \{1,2,3,\ldots,r+3\}$. Therefore, the weak set-labeling number of the wheel graph G is $3 + \lfloor \frac{n}{2} \rfloor$. □

Figure 4 illustrates the wheel graph W_7 with weak set-labeling number 6.

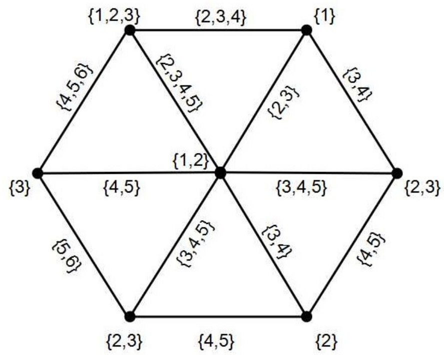

Fig. 4. A wheel graph with weak set-labeling number 6.

A *helm graph*, (see [3]) denoted by H_n, is a graph obtained by attaching a pendant edge to each vertex of the outer cycle of a wheel graph W_{n+1}. Then, the helm graph H_n has $2n+1$ vertices and $3n$ edges. That is, $H_n = W_{n+1} \odot K_1$, where $\odot$ is the corona of two graphs. The following theorem establishes the weak set-labeling number of the helm graph H_n.

THEOREM 3.2. *The weak set-labeling number of a helm graph H_n is $n+3$.*

PROOF. Let G be a helm graph on $2n+1$ vertices. Then, $G = (C_n+K_1) \odot K_1$. Let $\{v\}$ be the central vertex, $\{v_1,v_2,v_3,\ldots,v_n\}$ be the vertices in the cycle C_n of G and let $\{u_1,u_2,u_3,\ldots,u_n\}$ be the pendant vertices of G such that the vertex u_i is adjacent to the vertex v_i, $1 \leq i \leq n$. Since a vertex cover of H_n contains n elements, by Theorem 1.5, with respect to any WIASL defined on G, there must be at least n vertices in G must have singleton set-labels. Now, define an IASL f on G as follows. Label the vertices of C_n by $f(v_i) = \{i\}$, where $1 \leq i \leq n$. Then, label the pendant vertices of G such that $f(u_n) = \{1,2\}, f(u_{n-1}) = \{2,3\}, f(u_{n-2}) = \{3,4\}, \ldots, f(u_2) = \{n-1,n\}, f(u_1) = \{1,n\}$. The only vertex that remains to be labeled is the central vertex v. Since all subsets of X of the form $\{i,i+1\}$ have been used for labeling other vertices, choose the set $\{1,3\}$ to label the vertex v. Then,

$f(V(G)) \cup f^+(E(G)) = \{1, 2, 3, \ldots, n+3\}$. Since the minimal ground set X is $f(V(G)) \cup f^+(E(G))$, then $\sigma^*(G) = n+3$. □

Figure 4 illustrates the helm graph on 13 vertices with weak set-labeling number 7.

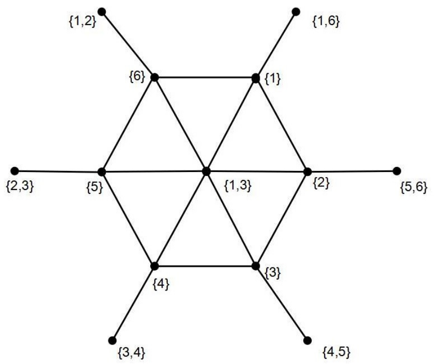

Fig. 5. A helm graph with weak set-labeling number 9.

A *friendship graph* F_n is a graph obtained by identifying one end vertex of n triangles. A friendship graph has $2n+1$ vertices and $3n$ edges. The following theorem determines the weak set-labeling number of a friendship graph.

THEOREM 3.3. *The weak set-labeling number of a friendship graph* F_n *is* $n+3$.

PROOF. Let v be the central vertex of the friendship graph F_n. Divide the remaining vertices of F_n in to two sets $U = \{u_1 u_2, u_3, \ldots, u_n\}$ and $W = \{w_1, w_2, w_3, \ldots, w_n\}$ such that the graph induced by the vertices v, u_i, w_i is a triangle in F_n, where $1 \leq i \leq n$. Let f be an IASL defined on F_n with respect to which, the set-labeling of the vertices of F_n is done in the following way. Let $f(v) = \{1\}$ and $f(u_i) = \{i+1\}$, where $1 \leq i \leq n$. Now, label the set W in such a way that $f(w_n) = \{1, 2\}, f(w_{n-1}) = \{2, 3\}, \ldots, f(w_2) = \{n-1, n\}, f(w_1) = \{n, n+1\}$. Then, $f^+(E) = \{2, 3, 4, \ldots, n+3\}$. Hence, the minimal ground set is $f(V) \cup f^+(E) = \{1, 2, 3, \ldots, n+3\}$ and $\sigma^*(G) = n+3$. □

Figure 6 illustrates a friendship graph F_4 with weak set-labeling number 7.

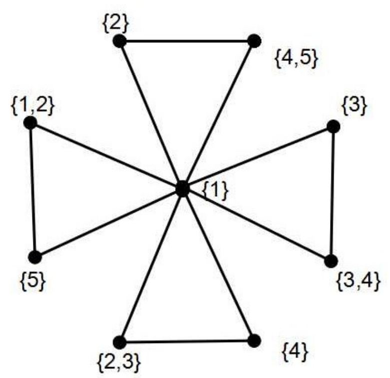

Fig. 6. A friendship graph with weak set-labeling number 7.

A *sunlet graph* SL_n is a graph obtained by attaching a pendant edge to all vertices of a cycle C_n (see [13]). The sunlet graph SL_n has $2n$ vertices and $2n$ edges. The following theorem establishes the weak set-labeling number of a sunlet graph.

THEOREM 3.4. *The weak set-labeling number of a sunlet graph* $SL_n = C_n \odot K_1$ *is* $n+2$.

PROOF. We have $G = C_n \odot K_1$. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of C_n and $\{u_1, u_2, u_3 \ldots, u_n\}$ be the pendant vertices of G. Let f be be an IASL defined on SL_n which assigns set-labels to the vertices of SL_n as follows. Let $f(u_n) = \{1, 2\}, f(u_{n-1}) = \{2, 3\}, f(u_{n-2}) = \{3, 4\}, \ldots, f(u_2) = \{n-1, n\}, f(v_1) = \{1, n\}$. Then, as explained in previous theorems, the minimal ground set $X = f(V(G)) \cup f^+(E(G)) = \{1, 2, 3, \ldots, n+2\}$. Hence, for the sunlet graph $G = C_n \odot K_1$, $\sigma^*(G) = n+2$. □

Another similar graph, that is considered here, is a sun graph which is defined as follows. A *sun graph* S_n is a graph obtained by replacing every edge of a cycle C_n by a triangle C_3 (see [2]). A sun graph also has $2n$ vertices. The same set-labeling, as explained in the previous theorem, can be applied to the vertices of a sun graph S_n and hence one can establish the following theorem.

THEOREM 3.5. *The weak set-labeling number of a sun graph* S_n *is* $n+3$.

PROOF. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of the cycle C_n and let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the independent vertices of S_n, such that the vertex u_i is adjacent to the vertices v_i and v_{i+1}, in the sense that $v_{n+1} = v_1$. As explained in the previous theorem, label the vertices of C_n as $f(v_i) = \{i\}$ and label the independent vertices such that $f(u_n) = \{1, 2\}, f(u_{n-1}) = \{2, 3\}, f(u_{n-2}) = \{3, 4\}, \ldots, f(u_2) = \{n-1, n\}, f(v_1) = \{1, n\}$. Then, $f^+(E(S_n)) = \{2, 3, 4, \ldots, n+3\}$. Hence, the minimal ground set $X = f(V) \cup f^+(E) = \{1, 2, 3, \ldots, n+3\}$. Therefore, the weak set-labeling number of the graph S_n is $n+3$. □

Figure 7 illustrates Theorem 3.5.

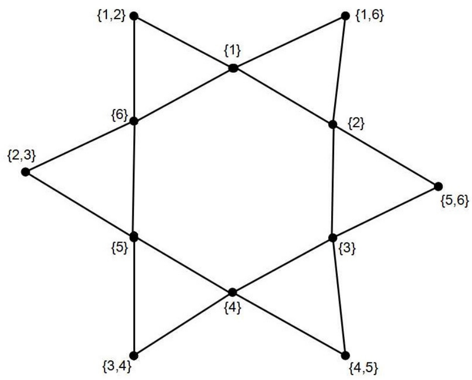

Fig. 7. A sun graph with weak set-labeling number 6

The next graph to study next for weak set-labeling number is a complete sun graph which is defined as follows. A *sun* (or *trampoline*) (see [2]) is a G on n vertices for some $n > 3$ whose vertex set can be partitioned into two sets, $W = \{w_1, w_2, w_3, \ldots, w_n\}$, $U = \{u_1, u_2, u_3, \ldots, u_n\}$, such that W is independent and for each i and j, the vertex w_i is adjacent to the vertices v_i and v_{i+1}, in the sense that $v_{n+1} = v_1$. A *complete sun* is a sun G in which the induced subgraph $\langle U \rangle$ is a complete graph. The following theorem determines the weak set-labeling number of a complete sun graph. A sun (or complete sun) has $2n$ vertices.

THEOREM 3.6. *The weak set-labeling number of a complete sun graph on $2n$ vertices is $n + 3$.*

PROOF. Let G be a complete graph on $2n$ vertices whose vertex sets are partitioned in to two sets U and W, where the induced subgraph $\langle U \rangle$ of G is a complete graph and W is an independent set. As usual, label the vertices of C_n as $f(u_i) = \{i\}$ and label the vertices in W in such a way that $f(w_n) = \{1, 2\}, f(w_{n-1}) = \{2, 3\}, f(w_{n-2}) = \{3, 4\}, \ldots, f(w_2) = \{n - 1, n\}, f(w_1) = \{1, n\}$. Then, $f^+(E) = \{2, 3, 4, \ldots, n+3\}$ and $f(V) \cup f^+(E) = \{1, 2, 3, \ldots, n + 3\}$. Hence, the minimum required cardinality for the ground set X is $\sigma^*(G) = |f(V) \cup f^+(E)| = n + 3$. □

Figure 8 illustrates a complete sun graph with a weak set-labeling number 7.

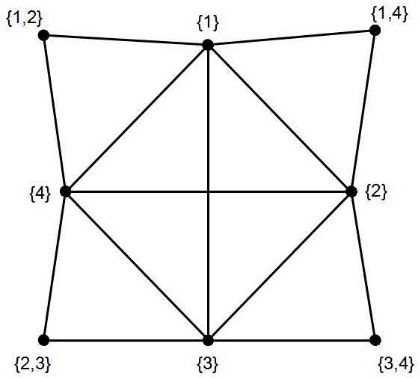

Fig. 8. A complete sun graph with weak set-labeling number 7

4. SCOPE FOR FURTHER STUDIES

In this paper, the discussion was on the weak set-labeling number and weak set-indexing number of certain classes of graphs. Certain Problems in this area are still open. Some of the open problems that identified in this area are the following.

PROBLEM 4.1. Determine the weak set-labeling number of a bipartite graphs, both regular and biregular.

PROBLEM 4.2. Determine the weak set-labeling number of certain graphs like gear graphs, lobsters, double wheel graphs etc.

PROBLEM 4.3. Determine the weak set-labeling number of certain graphs like double helm graphs, web graphs, windmill graphs etc.

PROBLEM 4.4. Determine the weak set-labeling number of split graphs, complete split graphs, bisplit graphs etc.

PROBLEM 4.5. Determine the weak set-labeling number of arbitrary sun graphs, rising sun graphs and partial sun graphs etc.

PROBLEM 4.6. Determine the weak set-labeling number of graphs, which admit a 1-uniform IASL.

Analogous to the weak set-labeling number of graphs, the notion of the weak set-indexing number of a graph G can be defined as follows.

DEFINITION 4.7. *The minimum cardinality of the ground set X, so that the function $f : V(G) \to \mathcal{P}(X)$ is a WIASI of G, is called the weak set-indexing number of G and is denoted by $\sigma^{\#}(G)$.*

Determining the weak set-indexing number of different graph classes is also an open problem.
There are several other types of standard graphs and named graphs whose weak set-labeling numbers and weak-set-indexing numbers can be calculated. Determining these parameters of different graph operations and graph parameters are also promising. Similar parameters corresponding to other types of integer additive set-labelings such as strong integer additive set-labelings, arithmetic integer additive set-labelings, exquisite integer additive set-labelings etc. are also worthy for future studies. All these facts show that there are wide scope for further studies in this direction.

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