New View of Ideals on PU-Algebra

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ABSTRACT
In this manuscript, we introduce a new concept, which called PU-algebra X . We state and prove some theorems about fundamental properties of it. Moreover, we give the concepts of a weak right self-maps, weak left self-maps and investigated some its properties. Further, we have proved that every associative PU-algebra is a group and every p-semisimple algebra is an abelian group. We define the centre of a PU-algebra X and show that it is a p-semisimple sub-algebra of X, which consequently implies that every PU-algebra contains a p-semisimple PU-algebra. Furthermore, we give the concepts of ideals ( -ideals, i=1,2,3,4) in PU-algebra , classified them into classes correspond to various formula and we have proved that, they are coincide . Mathematics Subject Classification: 06F35, 03G25, 08A30.

Keywords
PU-algebra, ideals of PU-algebra, G-part and P-radical of a PU-algebra, homomorphism of PU-algebra.

1. INTRODUCTION
In 1966, Imai and Iséki [2] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They are shown that the class of BCK-algebras is a proper subclass of the class of BCH-algebras. In [7], Négers and Kim introduced a notion of d-algebras, which is a generalization of BCK-Algebras and investigated a relation between d-algebras and BCK-algebras. Neggers et al. introduced the notion of Q-algebras [8], which is a generalization of BCH/BCI/BCK-algebras. Recently, Kim [3] defined a BE-algebra.[5] Meng, defined the notion of CI-algebra as a generalization of a BE-algebra.[4] Megalai and Tamilarasi introduced the notion of a TM-algebra which is a generalization of BCK/BCH/BCK/BCI-algebras and several results are presented. In 2009, C. Prabhayak and U. Leerawat [9,10] introduced algebraic structure which is called KU-algebras , and studied ideals and congruencies in KU-algebras. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. Moreover they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphisms and also investigated some of its properties. In this paper we will introduce a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated severed basic properties. Moreover we derived new view of several ideals on PU-algebra and studied some properties of them.

2. PRELIMINARIES
Now we will recall some known concepts related to PU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1[9]. By a KU-algebra we mean an algebra $(X, *, 0)$ of type $(2,0)$ with a single binary operation $*$ that satisfies the following identities:

$$(ku1): \ (x * y) * [(y * z) * (x * z)] = 0,$$

$$(ku2): \ x * 0 = 0,$$

$$(ku3): \ 0 * x = x ,$$

$$(ku4): \ x * y = 0 = y * x \ implies \ x = y.$$  

Example 2.2. Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
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It is easy to show that X is a KU-algebra.

Lemma 2.3 [6]. In a KU-algebra $(X, *, 0)$ , the following hold:

(i) $x \leq y$ imply $y * z \leq x * z$ .

(ii) $z * (y * x) = y * (z * x)$ .

Definition 2.4. A PU-algebra is a non-empty set $X$ with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(I) $0 * x = x ,$

(II) $(x * z) * (y * z) = y * x$ for any $x, y, z \in X .$

On $X$ we can define a binary relation $\leq$ by: $x \leq y$ if and only if $y * x = 0 .$

Example 2.5. Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by

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Using the algorithms in Appendix, we can prove that $(X, *, 0)$ is a PU-algebra, but not a KU-algebra, since $1 * 0 = 4 \neq 0$ . On the other hand, in Example 2.2., X is a KU-algebra, but is not a PU-algebra since $(2 * 1) * (3 * 1) = 1 * 0 \neq 3 * 2 = 2$, which means that PU-algebra and KU-algebra are deferent.
Example 2.6. $(R, \ast, 0)$ where $\ast$ is defined by $x \ast y = y - x$ for all $x, y \in R$ is a PU-algebra.

**Proposition 2.7.** In PU-algebra $(X, \ast, 0)$ the following hold for all $x, y, z \in X$:

(a) $x \ast x = 0$.
(b) $(x \ast z) \ast z = x$.
(c) $x \ast (y \ast z) = y \ast (x \ast z)$.
(d) $x \ast (y \ast x) = y \ast 0$.
(e) $(x \ast y) \ast 0 = y \ast x$.
(f) If $x \leq y$, then $x \ast 0 = y \ast 0$.
(g) $(x \ast y) \ast 0 = (x \ast z) \ast (y \ast z)$.
(h) $x \ast y \leq z$ if and only if $z \ast y \leq x$.
(i) $x \leq y$ if and only if $y \ast z \leq x \ast z$.
(j) In PU-algebra $(X, \ast, 0)$, the following are equivalent:

1. $x = y$.
2. $x \ast z = y \ast z$.
3. $z \ast x = z \ast y$.

(k) The right and the left cancellation laws hold in $X$.

**Proof:**

(a) Putting $x = y = 0$ in Definition 2.4. (II), we get $(0 \ast z) \ast (0 \ast z) = 0 \ast 0$. Then

$$z \ast z = 0$$
(by Definition 2.4. (I)).

(b) $(x \ast z) \ast z = (x \ast z) \ast (0 \ast z)$
(by Definition 2.4. (I))

$$= 0 \ast x = x$$
(by Definition 2.4. (I), (II)).

(c) $x \ast (y \ast z) = [(x \ast z) \ast z] \ast (y \ast z)$
(from (b))

$$= y \ast (x \ast z)$$
(by Definition 2.4. (II)).

(d) $x \ast (y \ast x) = y \ast (x \ast x)$
(from Proposition 2.7 (c))

$$= y \ast 0$$
(by Proposition 2.7 (a)).

(e) $(x \ast y) \ast 0 = (x \ast y) \ast (y \ast y)$
(from Proposition 2.7 (a))

$$= y \ast x$$
(by Definition 2.4. (II)).

(f) $x \leq y \Rightarrow y \ast x = 0$
(by the definition of PU-algebra)

$$x \ast 0 = x \ast (y \ast x) = y \ast 0$$
(from Proposition 2.7 (d)).

(g) $(x \ast y) \ast 0 = y \ast x$
(from Proposition 2.7 (e))

$$= (x \ast z) \ast (y \ast z)$$
(by Definition 2.4. (II)).

(h) $x \ast y \leq z \iff (x \ast y) \ast 0 \iff x \ast (z \ast y) = 0$
(from Proposition 2.7 (c))

$$\iff z \ast y \leq x$$.

(i) $x \leq y \iff y \ast x = 0$
(by the definition of PU-algebra)

$$\iff (x \ast z) \ast (y \ast z) = 0$$
(by Definition 2.4. (II))

$$\iff y \ast z \leq x \ast z$$.

(j) $(1) \Rightarrow (3)$: Clear.

$$(3) \Rightarrow (2))$: $z \ast x = z \ast y \Rightarrow (x \ast z) \ast 0 = (y \ast z) \ast 0$
(from Proposition 2.7 (e))

$$\Rightarrow ((x \ast z) \ast 0) \ast 0 = ((y \ast z) \ast 0) \ast 0$
(by Proposition 2.7 (b)).

$$(2) \Rightarrow (1))$: $x \ast z = y \ast z \Rightarrow (x \ast z) \ast z = (y \ast z) \ast z$

$$\Rightarrow x = y$$(from Proposition 2.7 (b)).

(k) Follows directly from (j).

**Proposition 2.8.** If $(X, \ast, 0)$ is a PU-algebra, then for any $x, y, z \in X$,

1. $(z \ast x) \ast (z \ast y) = x \ast y$.
2. $(x \ast y) \ast z = (z \ast y) \ast x$.

**Proof:**

(1) By the definition of PU-algebra, we have that

$$z \ast x) \ast (z \ast y) = [(x \ast y) \ast (z \ast y)] \ast [0 \ast (z \ast y)] = 0 \ast (x \ast y) = x \ast y$$.

(2) $(x \ast y) \ast z = [z \ast (x \ast y)] \ast 0$
(from Proposition 2.7 (e))

$$= [x \ast (z \ast y)] \ast 0$
(from Proposition 2.7 (c))

$$= (z \ast y) \ast x$
(from Proposition 2.7 (c)).

**Lemma 2.9.** If $(X, \ast, 0)$ is a PU-algebra, then $(X, \leq)$ is a partially ordered set.

**Proof:** By Proposition 2.7. (a), we have that $x \ast x = 0$ i.e. $x \leq x$.

Let $x \leq y, y \leq x$, then $x \ast y = 0 = y \ast x$. It follows that

$$x = 0 \ast x$$
(by Definition 2.4. (I))

$$= (y \ast y) \ast (0 \ast x) = 0 \ast y$$
(by Definition 2.4. (II).)

Let $x \leq y, y \leq z$ i.e. $x \ast y = 0 = z \ast y$. It follows that

$$z \ast x = 0 \ast (z \ast x)$$
(by Definition 2.4. (I))

$$= (y \ast x) \ast (z \ast x) = z \ast y$$
(by Definition 2.4. (II))

i.e. $x \leq z$. Therefore $(X, \leq)$ is a partially ordered set.

**Remark 2.10.** Every PU-algebra $(X, \ast, 0)$ satisfying $(y \ast x) \ast x = y \ast x$ for all $x, y \in X$ is a trivial algebra.

**Proof:** Putting $x = y$ in the equation $(y \ast x) \ast x = y \ast x$, we have $0 \ast x = 0$. By Definition 2.4. (I), $x = 0$. Hence $X$ is a trivial algebra.

**Proposition 2.11.** If $(X, \ast, 0)$ is a PU-algebra, then $(x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u)$ for all $x, y, z$ and $u \in X$.

**Proof:** Let $(X, \ast, 0)$ be a PU-algebra, then for all $x, y, z$ and $u \in X$ we have that

$$(x \ast y) \ast (z \ast u) = 0 \ast [(x \ast y) \ast (z \ast u)]$$
(by Definition 2.4. (I))

$$= [(y \ast u) \ast (y \ast u)] \ast [(x \ast y) \ast (z \ast u)]$$
(by Proposition 2.7 (a)).
\[(a) \text{ y } \in \text{G}(X) \implies x \ast (y \ast x) = y,
(b) \text{ y } \in \text{B}(X) \implies x \ast (y \ast x) = 0,
(c) \text{ x } \in \text{G}(X) \implies x \ast 0 = x
\]

Proof: (a) By Proposition 2.7, (d), \(x \ast (y \ast x) = y \ast 0 = y \implies y \in \text{G}(X)\).
(b) By Proposition 2.7, (d), \(x \ast (y \ast x) = y \ast 0 = 0 \implies y \in \text{B}(X)\).
(c) \(x \in \text{G}(X) \implies x \ast 0 = x \)
(by the definition of \(\text{G}(X)\))
\[\implies x \ast 0 = (x \ast 0) \ast 0 \quad \text{(by Proposition 2.7 (b))}
\implies x \ast 0 \in \text{G}(X).
\]

**Proposition 3.3.** The following are equivalent in \(\text{PU}(X, *, 0)\):

\(a) x \ast y = z, \quad b) y = z \ast x, \quad c) x = y \ast z, \quad d) x \ast y = z \ast x, \quad e) y \ast z = x \ast y, \quad f) x \ast (y \ast z) = (x \ast y) \ast z, \quad g) y \ast (x \ast z) = (y \ast x) \ast z, \quad h) x \ast (y \ast z) = (x \ast y) \ast z, \quad i) y \ast (x \ast z) = (y \ast x) \ast z, \quad j) x \ast (y \ast z) = (x \ast y) \ast z, \quad k) y \ast (x \ast z) = (y \ast x) \ast z, \quad l) x \ast (y \ast z) = (x \ast y) \ast z, \quad m) y \ast (x \ast z) = (y \ast x) \ast z, \quad n) x \ast (y \ast z) = (x \ast y) \ast z, \quad o) y \ast (x \ast z) = (y \ast x) \ast z, \quad p) x \ast (y \ast z) = (x \ast y) \ast z, \quad q) y \ast (x \ast z) = (y \ast x) \ast z, \quad r) x \ast (y \ast z) = (x \ast y) \ast z, \quad s) y \ast (x \ast z) = (y \ast x) \ast z, \quad t) x \ast (y \ast z) = (x \ast y) \ast z, \quad u) y \ast (x \ast z) = (y \ast x) \ast z, \quad v) x \ast (y \ast z) = (x \ast y) \ast z, \quad w) y \ast (x \ast z) = (y \ast x) \ast z, \quad x) x \ast (y \ast z) = (x \ast y) \ast z, \quad y) y \ast (x \ast z) = (y \ast x) \ast z, \quad z) x \ast (y \ast z) = (x \ast y) \ast z, \quad \\
(3) z = x \ast y \text{ for all } x, y, z \in \text{G}(X).
\]

**Proof:**
(1) \(\implies (2); x = y \ast z \implies z \ast x = x \ast (y \ast z) = y \ast 0 = y \)
(by Proposition 2.7 (d) and the definition of \(\text{G}(X)\)).
(2) \(\implies (3); y = z \ast x \implies x \ast y = x \ast (z \ast x) = z \ast 0 = z.
(3) \(\implies (1); z = x \ast y \implies y \ast z = y \ast (x \ast y) = x \ast 0 = x.

**Lemma 3.4.** If \(\text{G}(X) = X\), then \(X\) is \(P\)-semisimple.

Proof: Assume that \(\text{G}(X) = X\). Then \(\{0\} = \text{G}(X) \cap \text{B}(X) = X \cap \text{B}(X) = \text{B}(X)\), and hence \(X\) is \(P\)-semisimple.

**Definition 3.5.** Let \((X, \ast, 0)\) be a \(\text{PU}\)-algebra. For a fixed \(a \in X\).

The map \(\text{Ra}: X \to X\) given by \(\text{Ra}(y) = y \ast a\) for all \(y \in X\) is called a right \(\text{self-maps}\) of \(X\).

Similarly the map \(\text{Ma}: X \to X\) given by \(\text{Ma}(y) = (a \ast a) \ast (a \ast y)\) for all \(y \in X\) is called a left \(\text{self-maps}\) of \(X\).

**Theorem 3.7.** Let \((X, \ast, 0)\) be a \(\text{PU}\)-algebra, then \(Lx = Mx \circ Lx\) if and only if \(x \ast 0) \ast (x \ast (x \ast y)) = x \ast y \ast x \in X\).

**Corollary 2.12.** If \((X, \ast, 0)\) is a \(\text{PU}\)-algebra, then \((x \ast y) \ast (x \ast y) = (x \ast y) \ast 0 = 0 \ast 0 = 0\).

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x * (y * z) = (x * 0) * (y * z) = (x * y) * (0 * z) = (x * y) * z (by Definition 2.4(I)).

(b) By Definition 2.4(I) and the definition of G(X), we have
   x * y = (0 * x) * (y * 0) = (0 * y) * (x * 0) (by Proposition 2.11)
   = y * x (by Definition 2.4(I) and the definition of G(X)).

Theorem 3.11. If (X, *, 0) is a PU-algebra, then G(X) is an abelian group.

Proof: Let (X, *, 0) be a PU-algebra. Then for all x ∈ G(X) we have
   x = 0 * x = x * 0. By Proposition 2.7(a), we have x * x = 0 for all x ∈ G(X).
   Finally by Lemma 3.10(a), we have x * (y * z) = (x * y) * z for all x, y, z ∈ G(X).
   Therefore G(X) is an abelian group.

In Example 2.5, (X, *, 0) is a PU-algebra, but associatively does not hold, since
1 * (2 * 1) = 2 * 0 = 3 ≠ 0 = 2 * 0 = 1.

Theorem 3.12. If (X, *, 0) is a associative PU-algebra, then
G(X) = X and B(X) = {0}.

Proof: If (X, *, 0) is associative PU-algebra, then clearly G(X) ⊆ X.
   If x ∈ X, then x * x = x * 0 = x = 0 * x = x, and it follows that x ∈ G(X).
   Hence, G(X) ⊆ X. For the second part, clearly {0} ⊆ B(X).
   Thus G(X) = X. For the second part, clearly {0} ⊆ B(X).
   If x ∈ B(X), then x = 0 * x = (x * x) * x = x * x = 0 = 0.
   Thus B(X) = {0}.

Theorem 3.13. Every associative PU-algebra (X, *, 0) is a group.

Proof: Putting x = y = z in the associative law (x * y) * z = x * (y * z) and using Definition 2.4(I) and Proposition 2.7(a), we obtain 0 = x * x = x * 0.
   This means that 0 is the identity of X. Also by Proposition 2.7(a), every element x of X has an inverse. Therefore (X, *) is a group.

4. NEW VIEW OF IDEALS ON PU-ALGEBRA

Definition 4.1. A non-empty subset I of a PU-algebra (X, *, 0) is called an ideal of X if for any x, y ∈ X,
   (i) 0 ∈ I,
   (ii) x * y, x ∈ I imply y ∈ I.

Definition 4.2. A non empty subset I of a PU-algebra X is called a KU-ideal of X if it satisfies the following conditions:
   (1) 0 ∈ I,
   (2) x * (y * z) ∈ I, y ∈ I imply x * z ∈ I, for all x, y, z ∈ X.

Theorem 4.3. Let (X, *, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a KU-ideal of X.

Proof: (⇐): Suppose that I is an ideal of X. It is clear that 0 ∈ I. Let x * y ∈ I, y ∈ I imply x * z ∈ I. By the definition of PU-algebra, we have x * (y * z) = y * z and 0 * z = z, i.e., y * z ∈ I, y, z ∈ I imply z ∈ I. Therefore I is an ideal of X.

Example 4.4. Let X = {0, a, b, c} in which * is defined by the following table:

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<tr>
<th>*</th>
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<th>a</th>
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Using the algorithms in Appendix, we can prove that (X, *, 0) is a PU-algebra. It is easy to show that I2 = {0, a}, I2 = {0, b}, I3 = {0, c} are KU-ideals of X.

Definition 4.5. A non-empty subset I of a PU-algebra (X, *, 0) is called a PU1-ideal of X if it satisfies the following conditions:
   (i) 0 ∈ I,
   (ii) x * y, x * z ∈ I imply y * z ∈ I, for all x, y, z ∈ X.

Theorem 4.6. Let (X, *, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a PU1-ideal of X.

Proof: (⇒): Suppose that I is an ideal of X. It is clear that 0 ∈ I. Let y * x, x * z ∈ I. Since y * x = (x * z) * (y * z) (by Definition 2.4(II)), then we have (x * z) * (y * z) ∈ I and x * z ∈ I. It follows by the definition of ideal that y * z ∈ I. Therefore I is a PU1-ideal of X.

(⇐): Suppose that I is a PU1-ideal of X. It is clear that 0 ∈ I. Put y = 0 in the definition of PU1-ideal and by using the definition of PU-algebra, we get x ∈ I, x ∈ I implies z ∈ I. Therefore I is an ideal of X.

Definition 4.7. A non-empty subset I of a PU-algebra (X, *, 0) is called a PU2-ideal of X if for any x, y, z ∈ X,
   (i) 0 ∈ I,
   (ii) x * y, x * z ∈ I imply y * z ∈ I.

Theorem 4.8. Let (X, *, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a PU2-ideal of X.

Proof: (⇐): It is clear that 0 ∈ I. Let (x * y) * z ∈ I, z * y ∈ I. Since (X, *, 0) is a PU-algebra, then (z * y) * x = (x * y) * z ∈ I, it follows by the definition of an ideal of PU-algebra that x ∈ I. Hence I is a PU2-ideal of X.

(⇒): It is clear that 0 ∈ I. Let x * y ∈ I, x ∈ I. It follows by the definition of PU-algebra and its properties that x * y = (y * x) * 0 ∈ I and x = 0 * x ∈ I. Since I is a PU2-ideal of a PU-algebra, then y ∈ I. Hence I is an ideal of X.
Definition 4.9. A non-empty subset I of a PU-algebra \((X, \cdot, 0)\) is called a PU3-ideal of \(X\) if,

(i) \(0 \in I\),

(ii) \((a \cdot (b \cdot x)) \cdot x \in I\) for all \(a, b \in I\) and \(x \in X\).

Theorem 4.10. Let \((X, *, 0)\) be a PU-algebra and let \(I\) be a non-empty subset of \(X\). Then \(I\) is a PU3-ideal of \(X\) if and only if \(I\) is a PU1-ideal of \(X\).

Proof: Let \(I\) be a PU3-ideal of \(X\), obviously \(0 \in I\). Let \(x \neq y, y \neq z \in I\). Now applying (Definition 2.4, (I), (II)), we get

\[
x \cdot z = 0 \cdot (x \cdot z) = \left\{ 6 \cdot 4 \cdot 4 \cdot 4 \cdot 7 \cdot 7 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 8 \cdot 8 \right\} \cdot (x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) \cdot (x \cdot z) = a \cdot b \cdot (x \cdot z) \cdot (x \cdot z)^\cdot (x \cdot z)
\]

Hence \(I\) is a PU1-ideal of \(X\).

Conversely, if \(I\) is a PU1-ideal of \(X\), it is clear that \(0 \in I\) and (by Theorem 4.6) \(I\) is an ideal of \(X\).

To prove (ii) of Definition 4.9, observe that \((a \cdot (b \cdot x)) \cdot x \in I\) if and only if \((a \cdot (b \cdot x)) \in I\) for all \(a, b \in I\) and \(x \in X\). By Proposition 2.7 (c), we have a \(a \cdot (b \cdot (x \cdot z)) \in I\) for all \(a, b \in I\) and \(x \in X\). By Proposition 2.7 (c), we have \(b \cdot (a \cdot ((b \cdot x)) \in I\). Since \(I\) is an ideal and \(a \in I\), it follows that \((a \cdot (b \cdot x)) \in I\). By Proposition 2.7 (c), we have \((b \cdot (a \cdot (b \cdot x)) \in I\). Since \(I\) is an ideal and \(b \in I\), it follows that \((a \cdot (b \cdot x)) \in I\). Therefore \(I\) is a PU3-ideal of \(X\).

Lemma 4.11: If \(I\) is a PU3-ideal of a PU-algebra \(X\), then for every \(a \in I\) and \(x \in X\),

\[(a \cdot x) \cdot x \in I\]

Proof: Clear.

Corollary 4.12: If \(a \in I\) and \(x \leq a\), then \(x \in I\).

Proof: The condition \(x \leq a\) in a PU-algebra means \(a \cdot x = 0\) and by Lemma 4.11, we get \(x = 0 = 0 \cdot x = (a \cdot x) \cdot x \in I\).

Definition 4.13. A non-empty subset \(I\) of a PU-algebra \((X, *, 0)\) is called a PU4-ideal of \(X\) if,

(i) \(0 \in I\),

(ii) \((a \cdot 0) \cdot b \in I\) for all \(a, b \in I\).

Lemma 4.14. If \((X, *, 0)\) is a PU-algebra, then \((x \cdot (y \cdot z)) \cdot z = (y \cdot z) \cdot x\) for all \(x, y, z \in X\).

Proof: Let \((X, *, 0)\) be a PU-algebra and let \(x, y, z \in X\), then we have that \((x \cdot (y \cdot z)) \cdot z = (y \cdot z) \cdot x\) (by Proposition 2.8 (2))

\[= (y \cdot z) \cdot x \quad (\text{by Proposition 2.7 (c)})
\]

Theorem 4.15. Let \((X, *, 0)\) be a PU-algebra and let \(I\) be a non-empty subset of \(X\). Then \(I\) is a PU3-ideal of \(X\) if and only if \(I\) is a PU4-ideal of \(X\).


The following result is a direct consequence of Theorems (4.3, 4.6, 4.8, 4.10 and 4.15).

Theorem 4.16. If \(X\) is a PU-algebra, then the following are equivalent:

1) \(I\) is an ideal of \(X\).

2) \(I\) is a KU-ideal of \(X\).

3) \(I\) is a PU1-ideal of \(X\).

4) \(I\) is a PU2-ideal of \(X\).

5) \(I\) is a PU3-ideal of \(X\).

6) \(I\) is a PU4-ideal of \(X\).

Lemma 4.17. Let \((X, *, 0)\) be a PU-algebra and \(\{A_i\}_i\) be a family of PU1-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also PU1-ideal of \(X\).

Proof: Let \(x, y, z \in X\) be such that \(y \cdot z \in A_i \) for all \(i \in I\). But \(A_i \) is a PU1-ideal of \(X\) if \((y \cdot z) \in A_i \). Therefore \(\bigcap_{i \in I} A_i \) is also PU1-ideal of \(X\).

Remark 4.18. Let \((X, *, 0)\) be a PU-algebra.

1) If \(\{A_i\}_i\) is a family of KU-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also KU-ideal of \(X\).

2) If \(\{A_i\}_i\) is a family of PU1-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also PU1-ideal of \(X\).

3) If \(\{A_i\}_i\) is a family of PU2-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also PU2-ideal of \(X\).

4) If \(\{A_i\}_i\) is a family of PU3-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also PU3-ideal of \(X\).

5) If \(\{A_i\}_i\) is a family of PU4-ideals of \(X\), then \(\bigcap_{i \in I} A_i \) is also PU4-ideal of \(X\).

Proposition 4.19. If \((X, *, 0)\) is a PU-algebra, then

(a) \(G(X)\) is a PU1-ideal of \(X\).

(b) \(B(X)\) is a PU1-ideal of \(X\).

Proof: (a) Clearly \(0 \not\in G(X)\). Let \(x \cdot y \in G(X), x \in G(X)\). Then We have that

\[y \cdot 0 = x \cdot (y \cdot x) \quad (\text{by Proposition 2.7 (d)})
\]

\[= x \cdot ((x \cdot y) \cdot 0) \quad (\text{by Proposition 2.7 (e)})
\]

\[= x \cdot (x \cdot y) \quad (\text{by the definition of G(X)})
\]

Since \(G(X) = X\) is a PU-sub algebra of \(X\), then \(y \cdot 0 \in G(X)\). Hence by Proposition 2.7 (b), we have that \(y \cdot 0 = (y \cdot 0) \cdot 0 = y\). Then \(y \cdot 0 = y\) and \(G(X)\) is an ideal of \(X\). Therefore by Theorem 4.6, we have that \(G(X)\) is a PU1-ideal of \(X\).

(b) Clearly \(0 \in B(X)\). Let \(x \cdot y \in B(X), x \in B(X)\). We have that

\[y \cdot 0 = x \cdot (y \cdot x) = x \cdot ((x \cdot y) \cdot 0) \quad (\text{by Proposition 2.7 (d),(e)})
\]

\[= x \cdot 0 = 0 \quad (\text{by the definition of B(X)})
\]
Then $y \in B(X)$, and thus $B(X)$ is an ideal of $X$. Therefore by Theorem 4.6, we have that $B(X)$ is a PU1-ideal of $X$.

5. HOMOMORPHISMS OF PU-ALGEBRA

**Definition 5.1.** Let $(X, *, 0)$ and $(X', \star, \emptyset, 0')$ be PU-algebras. A map $f: X \rightarrow X'$ is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

**Theorem 5.2.** Let $(X, *, 0)$ and $(X', \star, \emptyset, 0')$ be PU-algebras, and if $f: X \rightarrow X'$ be a homomorphism, then

(1) $f(0) = 0'$.

(2) If $S$ is a PU-subalgebra of $X$, then $f(S)$ is a PU-subalgebra of $X'$.

(3) If $S$ is a PU-subalgebra of $X$, then $f^{-1}(S)$ is a PU-subalgebra of $X$.

(4) If $x \leq y$, then $f(x) \leq f(y)$.

(5) If $B$ is a PU1-ideal of $X$, then $f^{-1}(B)$ is a PU1-ideal of $X$.

(6) ker $f$ is a PU1-ideal of $X$.

**Proof:**

(1) $f(0) = f(0 \ast 0) = f(0) \ast f(0) = 0'$ (by Definition 2.4. (i), Definition 5.1. and Proposition 2.7. (a)).

(2) Let $x \ast y \in f(S)$. It follows that $x \ast y = f(x) \ast f(y)$ for some $x, y \in S$. It follows by Definition 5.1., that $x \ast y = f(x) \ast f(y) = f(x \ast y)$. Since $S$ is a PU-subalgebra of $X$, then $x \ast y \in S$ and hence $x \ast y \in f(S)$ which complete the proof.

(3) Let $x, y \in f^{-1}(S)$. It follows that $f(x), f(y) \in S$. Since $S$ is a PU-subalgebra of $X$ and $f$ is a homomorphism, then $f(x) \ast f(y) = f(x \ast y) \in f(S)$ which complete the proof.

(4) Since $x \leq y$, then $y \ast x = 0$. It follows that $f(y \ast x) = f(0) = 0'$. Since $f$ is a homomorphism, then $f(y) \ast f(x) = f(0) = 0'$. Therefore $f(x) \leq f(y)$.

(5) Since $B$ is a PU1-ideal of $X$, then $0 \in B$ (i.e. $f(0) \in B$). It follows that $0 \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $x \ast y \in f^{-1}(B), x \ast z \in f^{-1}(B)$. It follows that $f(x \ast y \ast z) \in B$. Since $f$ is a homomorphism, then $f(x) \ast f(y) \ast f(z) \in B$. Since $B$ is a PU1-ideal of $X$, then $f(x) \ast f(y) \ast f(z) \in B$. Since $f$ is a homomorphism, then $f(x \ast y \ast z) \in B$. It follows that $y \ast z \in f^{-1}(B)$. Therefore $f^{-1}(B)$ is a PU1-ideal of $X$.

(6) It is clear that $0 \in \ker f$. Let $x, y, z \in X$ be such that $x \ast y \in \ker f$, then $f(x \ast y) = 0'$. Since $S$ is a PU-subalgebra of $X$, then $y \ast x = 0'$. Since $f$ is a homomorphism, then $y \ast x = f(0) \ast f(x) = f(0) \ast f(x) = 0'$. It follows that $0 \ast f(y \ast x) = f(0) = 0'$ (i.e. $y \ast z \in \ker f$). Therefore $\ker f$ is a PU1-ideal of $X$.

6. CONCLUSION

In this manuscript, we introduce a new concept, which called PU-algebra $X$.

We state and prove some theorems about fundamental properties of it. Moreover, we give the concepts of a weak right self-maps, weak left self-maps and investigated some its properties. Further, we have proved that every associative PU-algebra is a group and every p-semisimple algebra is an abelian group. We define the centre of a PU-algebra $X$ and show that it is a p-semisimple sub-algebra of $X$, which consequently implies that every PU-algebra contains a p-semisimple PU-algebra.

We posed the following problem, is the set Hom(X) of all PU-homomorphisms of X into itself, is a PU-algebra? We can proved that it is not always a PU-algebra. However, it may be established that Hom(X) is a PU-algebra, if $X$ is an associative PU-algebra. But an associative PU-algebra is again a p-semisimple algebra. Thus homological study of PU-algebras did not develop for PU-algebras in general. The future purpose of this paper is to study the set of all left-regular self-maps of a positive implicative PU-algebra $X$, we can show that it forms a positive implicative PU-algebra. But no such effort was made for PU-algebras, We form weakly positive implicative PU-algebras in terms of its Right Self-maps and Weak Right Self-maps. Further, some properties of Weak Right Self-maps, Weak Left Self-maps and Weak Left-regular Self-maps can be studied. It can also shown that the set of all Weak Left-Regular Self-maps of a weakly positive implicative PU-algebra $X$, is a weakly positive implicative PU-algebra. Thus homological study has been made in the class of weakly positive implicative PU-algebras a class which contains the class of p-semisimple PU-algebras, the class of associative PU-algebra, the class of weakly positive implicative PU-algebras and weakly positive implicative PU-algebras. As is well known, the concept of ideal I plays an important role in PU-algebras. And a lot of results on ideals can be obtained. We have classified ideals into the following classes as follows: Ideals have elements of $X$, ideals have elements of $X$ and $I$ and Ideals have elements of $I$. We know that every ideal is not necessarily a sub-algebra. Thus a question arises - what type of ideals are sub-algebras? We hope in the further work can answer these open questions.

7. ACKNOWLEDGMENTS

The authors are greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

**Algorithms for PU-algebra**

Input: $(X)$ set with $0$ element; Binary operation $*$

Output: $("X is a PU-algebra or not")$

If $X = \emptyset$ then;
Go to (1.)
End if

If $0 \notin X$ then go to (1.);
End if
Stop: = false
i = 1;
While $i \leq |X|$ and not (Stop) do
If $0 \ast x \neq x$, then
Stop: = true
End if
j = 1;
While $j \leq |X|$ and not (Stop) do
k = 1;
While $k \leq |X|$ and not (Stop) do
If $(x \ast x) \ast (x \ast x) \neq x \ast x$, then
Stop: = true
End if
End while
End if
End while
If stop then

International Journal of Computer Applications (0975 – 8887)
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Output ("X is a PU-algebra")
Else
(1.) Output ("X is not a PU-algebra")
End if
End.

Algorithms for PU-ideal in PU-algebra
Input (X: PU-algebra, I: subset of X)
Output ("I is a PU-ideal of X or not")

If I = \emptyset then
Go to (1.);
End if
If 0 \notin I then
Go to (1.);
End if
Stop := false
i = 1;
While i \leq |X| and not (stop) do
j = 1
While j \leq |X| and not (stop) do
k = 1
While k \leq |X| and not (stop) do
If x_j * x_i \in I and x_i * x_k \in I then
If x_j * x_k \notin I then
Stop := false
End if
End while
End while
End while
If stop then
Output ("I is a PU-ideal of X")
Else

(1.) Output ("I is not ("I is a PU-ideal of X")
End if

8. REFERENCES