PH/PH/1 Bulk Arrival and Bulk Service Queue

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ABSTRACT
This paper studies two stochastic bulk arrivals and bulk services PH/PH/1 queue Models (A) and (B) with \( k_1 \) and \( k_2 \) as the number of phases of PH arrival and PH service distributions respectively. The system has infinite storing capacity and the arrival and service sizes are finite valued random variables. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes is greater than the maximum of the service sizes and the infinitesimal generator is partitioned as blocks of \( k_1 k_2 \) times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes is less than the maximum of the service sizes. The generator is partitioned using blocks of \( k_1 k_2 \) times the maximum of the service sizes. Block circulant matrix structure is noticed in the basic system generator. The stationary queue length probabilities, its expected values, its variances and probabilities of empty levels are derived for the two models using matrix geometric methods. Numerical examples are presented for illustration.

General Terms
Bulk Arrivals, Bulk Service, Block Circulant Matrix, Markov Chain and Phase Type Distribution

Keywords
Block Sizes, Stationary Probability, Infinitesimal Generator and Matrix Geometric Approach.

1. INTRODUCTION
In this paper two bulk arrival and bulk service PH/PH/1 queues have been studied using matrix geometric methods. Retrial queues are studied by Aissani.A and Artalejo.J.R [1] and Ayyappan, Subramanian and Gopal Sekar [2]. Numerical methods on matrix methods are presented by Bini, Latouche and Meini [3]. Multi server model has been of interest in Chakravarthy and Neuts [4]. Birth and death model has been analyzed by Gaver, Jacobs and Latouche [5]. Analytic methods are focused in Latouche and Ramaswami [6] and for matrix geometric methods one may refer Neuts [7]. The models considered here are general compared to existing models. Here random number of arrivals and random number of services are considered at a time whereas a fixed number of customers arrive or are served at any arrival or service epochs in existing queue models. Fixed numbers of customers are cleared by a service in the models of Neuts and Nadarajan [8]. In real life situations when a machine manufactures a fixed number of products in every production schedule, the defective items are rejected in all production lots, making the production lot is only of random size and not a fixed one always. Situations of random bulk services are seen often in software based industries where finished software projects waiting for marketing are sold in bulk sizes when there is economic boom and the business may be insignificant when there is economic recession. In industrial productions, bulk types are very common. Manufactured products arrive in various bulk sizes for sale in markets and the products are sold in various bulk sizes depending on market requirements. Noam Paz and Uri Yechali [9] have studied M/M/1 queue with disaster. Usually bulk arrival models have M/G/1 upper-Heisenberg block matrix structure. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector as done in William J. Stewart [10] and even in such models the recurrence relation method to find the stationary probabilities is stopped at certain level in most general cases indicating limitations of such approach. For M/M/1 bulk queues with random environment models one may refer Rama Ganesan, Ramshankar and Ramanarayanan [11]. In this paper the partitioning of the matrix is carried out in a way that the stationary probability vector exhibits a matrix geometric structure for PH/PH/1 bulk queues where the arrivals and service sizes are finite. Two models (A) and (B) on PH/PH/1 bulk queue systems with finite storage space for customers are studied here using the block partitioning method. In the models considered here, the maximum arrival sizes and the maximum service sizes are different. Model (A) presents the case when \( M \), the maximum of arrival sizes is bigger than \( N \), the maximum of the sales sizes. In Model (B), its dual case \( N \) is bigger than \( M \), is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of \( M \) sized blocks of customers when \( M > N \) and \( N \) sized blocks of customers when \( N > M \) for finding the rate matrix. Using the maximum of the bulk arrival size or the maximum of the bulk service size and grouping the customers as members of the blocks for the partitioning the infinitesimal generator is a new approach in this area. The matrices appearing as the basic system generator in these two models due to block partitioned structure are seen as a block circulant matrices. The stationary probability of the number of customers waiting for service, the expectation and the variance and the probability of empty queue are derived for these models. Numerical cases are presented to illustrate them.

2. MODEL (A): MAXIMUM ARRIVAL SIZE \( M > MAXIMUM SERVICE SIZE N \)

2.1 Assumptions
i) The time between consecutive epochs of bulk arrivals of customers has phase type distribution \((\alpha, T)\) where \( T \) is a matrix of order \( k_1 \) with absorbing rate \( T_{0} = -T_{e} \) to the absorbing state \( k_1 + 1 \) from where the arrival process moves instantaneously to a starting state as per the starting vector \( \varphi = (\alpha_1, \alpha_2, ..., \alpha_{k_1}) \) and \( \sum_{i=1}^{k_1} \alpha_i = 1 \). Let \( \varphi \) be the invariant probability vector of the generator matrix \((\varphi + T_{0} \varphi_{0})\).

ii) When the absorption occurs in the PH arrival process due to transition from a state \( i \) to state \( k_1 + 1 \), \( x_{i} \) number of customers arrive with probabilities \( P(x_{i} = j) = p_{i}^{j} \) for \( 1 \leq j \leq M_{i} \) and \( \sum_{i=1}^{N} M_{i} = 1 \) where \( M_{i} \) is the maximum size for \( 1 \leq i \leq k_{1} \).
iii) The time between consecutive epochs of bulk services of customers has phase type distribution \( (\beta, S) \) where \( S \) is a matrix of order \( k_2 \) with absorbing rate \( S_0 = -Se \) to the absorbing state \( k_2+1 \) from where the service process moves instantaneously to a starting state as per the starting vector \( \beta = (\beta_1, \beta_2, \ldots, \beta_{k_2}) \) and \( \sum_{i=1}^{k_2} \beta_i = 1 \). Let \( \phi \) be the invariant probability vector of the generator matrix \( (S + \lambda S_0) \).

iv) Customers of bulk size \( \psi_i \) are served at epochs when the absorption occurs due to a transition from state \( i \) to state \( k_2+1 \), with probabilities \( P(\psi_i = j) = q_{ij} \) for \( 1 \leq j \leq N_i \) and \( \sum_{j=1}^{N_i} q_{ij} = 1 \). When more than \( N_i \) customers are waiting for service where \( N_i \) is the maximum for \( 1 \leq i \leq k_1 \). When \( n \) customers \( n < N_i \) are waiting for service, then \( j \) customers are served with probability \( q_{ij} \), for \( 1 \leq j \leq n-1 \) and \( n \) customers are served with probability \( \sum_{j=1}^{n} q_{ij} \) for \( 1 \leq i \leq k_2 \).

v) The maximum arrival size \( M = \max(\psi_i) \leq k_1, M_i \) is greater than the maximum service size \( N = \max(\psi_i) \leq k_2, N_i \).

2.2. Analysis

The state of the system of the continuous time Markov chain \( X(t) \) under consideration is presented as follows. \( X(t) = \{(0, i) : 1 \leq i \leq k_1 \} \cup \{(0, k, i, j) : 1 \leq k \leq M; 1 \leq i \leq k_1; 1 \leq j \leq k_2 \} \cup \{(n, k, i, j) : 0 \leq k \leq M; 1 \leq i \leq k_1; 1 \leq j \leq k_2 \text{ and } n \geq 0 \} \).

The chain is in the state \((0, i)\) when the number of customers in the queue is 0, and the arrival phase is \( i \) for \( 1 \leq i \leq k_1 \). The chain is in the state \((0, k, i, j)\) when the number of customers \( k_1 + k_2 \) in \( k_1 \) and \( k_2 \) and the service phase is \( j \) for \( 1 \leq j \leq k_2 \). The chain is in the state \((n, k, i, j)\) when the number of customers in the queue is \( n \) \( k \) and \( k_2 \) and the service phase is \( j \) for \( 1 \leq j \leq k_2 \). When the number of customers waiting in the system is \( r \), then \( r \) is identified with \((n, k)\) where \( r \) on division by \( M \) gives \( n \) as the quotient and \( k \) as the remainder. Let the survivor probabilities of arrivals \( \lambda_i \) and of services \( \psi_i \) be respectively 

\[
P(\xi_t > m) = p_{m}^{\lambda} = \prod_{i=1}^{m} \lambda_i,\quad \text{for } 1 \leq m \leq M_1 \text{ and } 1 \leq i \leq k_1 \quad (2)
\]

\[
P(\psi_t > m) = q_{m}^{\alpha} = \prod_{i=1}^{m} q_i,\quad \text{for } 1 \leq m \leq N_i \text{ and } 1 \leq i \leq k_2 \quad (3)
\]

with \( p_0^{\lambda} = 1 \), for all \( i \), \( 1 \leq i \leq k_1 \) and \( q_0^\alpha = 1 \) for all \( i, 1 \leq i \leq k_2 \).

The chain \( X(t) \) describing model has the infinitesimal generator \( Q_A \) of infinite order which can be presented in block partitioned form given below.

\[
\begin{bmatrix}
A_M & 0 & \cdots & 0 & 0 & 0 \\
A_{M-1} & A_M & \cdots & 0 & 0 & 0 \\
A_{M-2} & A_{M-1} & \cdots & 0 & 0 & 0 \\
A_{M-3} & A_{M-2} & \cdots & \vdots & \vdots & \vdots \\
A_3 & A_4 & \cdots & A_M & 0 & 0 \\
A_2 & A_3 & \cdots & A_{M-1} & A_M & 0 \\
A_1 & A_2 & \cdots & A_{M-2} & A_{M-1} & A_M
\end{bmatrix}
\]

(8)

\[
Q_A = \begin{bmatrix}
B_1 & B_0 & 0 & 0 & \cdots & 0 \\
B_2 & A_1 & A_0 & 0 & \cdots & 0 \\
0 & A_2 & A_1 & A_0 & \cdots & 0 \\
0 & 0 & A_2 & A_1 & A_0 & \cdots \\
0 & 0 & 0 & A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(4)

In (4) the states of the matrices are listed lexicographically as \( 0, 1, 2, 3, \ldots \). For partition purpose the zero states in the first two sets of (1) are combined. The vector \( \varnothing \) is of type \( 1 \times [k_1 + k_2(M-1)] \) and the vector \( \eta \) is of type \( 1 \times (k_1k_2M) \).

\[
\begin{bmatrix}
(0,1,0,0) & (0,2,0,0) & \cdots & (0,k_1,0,0) \\
(0,0,1,1) & (0,0,1,2) & \cdots & (0,0,k_1,1,1) \\
(0,1,1,1) & (0,1,1,2) & \cdots & (0,1,k_1,1,1) \\
\vdots & \vdots & \vdots & \vdots \\
(0,3,1,1) & (0,3,1,2) & \cdots & (0,3,k_1,1,1) \\
(0,2,1,1) & (0,2,1,2) & \cdots & (0,2,k_1,1,1) \\
\vdots & \vdots & \vdots & \vdots \\
(0,1,k_1,1,1) & (0,1,k_1,1,2) & \cdots & (0,1,k_1,k_2,1,1) \\
(0,k_1,1,1) & (0,k_1,1,2) & \cdots & (0,k_1,k_2,1,1) \\
(0,k_1,k_2,1,1) & (0,k_1,k_2,1,2) & \cdots & (0,k_1,k_2,k_2,1,1)
\end{bmatrix}
\]

(5)

The matrices \( I \) and \( A_d \) have negative diagonal elements, they are of orders \( k_1 + k_2(M-1) \) and \( k_1k_2M \) respectively and their off diagonal elements are non-negative. The matrices \( A_0 \) and \( A_d \) have nonnegative elements and are of order \( k_1k_2M \). The matrices \( B_0 \) and \( B_2 \) have non-negative elements and are of type \( [k_1 + k_2(M-1)] \times (k_1k_2M) \) and they are given below. Let \( \mathcal{D} \) and \( \mathcal{S} \) denote the Kronecker sum and Kronecker products respectively. Let \( \varPhi = \mathcal{D} \mathcal{S} = (\mathcal{T} \otimes I_k) + (I_k \otimes \mathcal{S}) \).

Let \( Q_1 = \varPhi \varphi \) denote the invariant probability vector of absorption rates concerning the PH service distribution. Let \( S_{ij} = (s_i^1, s_i^2, \ldots, s_i^k)^T \) be the column vector of absorption rates in PH arrival process. Let \( S_0 = (s_0^1, s_0^2, \ldots, s_0^k)^T \) be the column vector of absorption rates concerning the PH service distribution.

\[
A_j = [T_{ij} \alpha] \otimes A_{k_i} \quad \text{for } 1 \leq j \leq M
\]

(6)

\[
U_j = I_k \otimes [S_0 \beta] \quad \text{for } 1 \leq j \leq M \quad \text{with orders } k_1 \times k_2
\]

\[
A_j = [T_{ij} \alpha] \otimes A_{k_i} \quad \text{for } 1 \leq j \leq M \quad \text{with orders } k_1 \times k_2
\]

(5)

\[
U_j = I_k \otimes \left[ (S_0^1 \beta_1, s_0^2 \beta_1, \ldots, s_0^k \beta_1)^T \right] \quad \text{for } 1 \leq j \leq M
\]

\[
U_j = I_k \otimes \left[ (S_0^1 \beta_1, s_0^2 \beta_2, \ldots, s_0^k \beta_2)^T \right] \quad \text{for } 1 \leq j \leq M
\]

(7)

The matrix \( B_0 \) is same as that of \( A_0 \) when \( A_M \) in \( A_0 \) is replaced by \( A_M' \). The matrix \( B_2 \) is same as that of \( A_2 \) when the first block column with \( 0 \) is considered as \( k_1 \) columns block instead of \( k_1 k_2 \) columns block of \( A_2 \).
\[
A_1 = \begin{bmatrix}
Q_1 & A_1 & A_2 & \ldots & A_{M-N-2} & A_{M-N-1} & A_{M-N} & \ldots & A_{M-2} & A_{M-1} \\
U_1 & Q_1 & A_1 & \ldots & A_{M-N-3} & A_{M-N-2} & A_{M-N-1} & \ldots & A_{M-3} & A_{M-2} \\
U_2 & U_1 & Q_1 & \ldots & A_{M-N-4} & A_{M-N-3} & A_{M-N-2} & \ldots & A_{M-4} & A_{M-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
U_N & U_{N-1} & U_{N-2} & \ldots & Q_1 & A_1 & A_2 & \ldots & A_{M-N-2} & A_{M-N-1} \\
0 & U_N & U_{N-1} & \ldots & U_1 & Q_1 & A_1 & \ldots & A_{M-N-3} & A_{M-N-2} \\
0 & 0 & U_N & \ldots & U_2 & U_1 & Q_1 & \ldots & A_{M-N-4} & A_{M-N-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & U_N & U_{N-1} & U_{N-2} & \ldots & Q_1 & A_1 \\
0 & 0 & 0 & \ldots & \ldots & U_N & U_{N-1} & \ldots & U_N & U_{N-1} \\
\end{bmatrix}
\] (10)

\[
B_1 = \begin{bmatrix}
T & A_1 & A_2 & \ldots & A_{M-N-2} & A_{M-N-1} & A_{M-N} & \ldots & A_{M-2} & A_{M-1} \\
U & Q_1 & A_1 & \ldots & A_{M-N-3} & A_{M-N-2} & A_{M-N-1} & \ldots & A_{M-3} & A_{M-2} \\
V_1 & U_1 & Q_1 & \ldots & A_{M-N-4} & A_{M-N-3} & A_{M-N-2} & \ldots & A_{M-4} & A_{M-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & U_N & U_{N-1} & \ldots & U_1 & Q_1 & A_1 & \ldots & A_{M-N-3} & A_{M-N-2} \\
0 & 0 & U_N & \ldots & U_2 & U_1 & Q_1 & \ldots & A_{M-N-4} & A_{M-N-3} \\
0 & 0 & 0 & \ldots & U_N & U_{N-1} & U_{N-2} & \ldots & Q_1 & A_1 \\
0 & 0 & 0 & \ldots & \ldots & U_N & U_{N-1} & \ldots & U_N & U_{N-1} \\
\end{bmatrix}
\] (11)

The basic generator of the bulk queue which is concerned with only the arrival and service is a matrix of order \(k_1k_2M\) given above in (12) where \(Q_0 = A_0 + A_1 + A_2\) (13).

Its probability vector \(w\) gives, \(wQ_0 = w\) and \(w.e = 1\) (14).

It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix \(Q_0 = A_0 + A_1 + A_2\) is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition.

In (13), the first block-row of type \(k \times kM\) is, \(W = (Q_1 + A_M, A_1, A_2, \ldots, A_{M-N-2}, A_{M-N-1}, A_{M-N} + U_i, \ldots, A_{M-2} + U_i, A_{M-1} + U_i)\) which gives as the sum of the blocks \((Q_1 + A_M) + A_1 + A_2 + \ldots + A_{M-N-2} + A_{M-N-1} + A_{M-N} + U_i + \ldots + U_i, \ldots + U_i, A_{M-2} + U_i, A_{M-1} + U_i) = (T + T_0Q) (S + S_0)\beta\) whose stationary vector of \(\varphi \otimes \varphi$. This gives \(\varphi \otimes \varphi (Q_1 + A_M) + \varphi \otimes \varphi (\sum_{i=1}^{M-1} A_i + \varphi \otimes \varphi (A_{M-N} + U_i)) = 0. \) So \((\varphi \otimes \varphi, \varphi \otimes \varphi, \ldots, \varphi \otimes \varphi) W = w = (\varphi \otimes \varphi, \varphi \otimes \varphi, \ldots, \varphi \otimes \varphi) W)\) Since all blocks, in any block-row are seen somewhere in each and every column block structure (the matrix is block circulant), the above equation shows the left eigen vector of the matrix \(Q_0\) is \((\varphi \otimes \varphi, \varphi \otimes \varphi, \ldots, \varphi \otimes \varphi)\). Using (14), this gives probability vector

\[
w = (\varphi \otimes \varphi, \varphi \otimes \varphi, \ldots, \varphi \otimes \varphi) W = (\sum_{i=1}^{M-1} \varphi \otimes \varphi) e = (\sum_{i=1}^{M-1} \varphi \otimes \varphi) e
\] (15)

Neuts [7], gives the stability condition as, \(wA_0 e < wA_2 e\) where \(w\) is given by (15). Taking the sum cross diagonally in the \(A_0\) and \(A_2\) matrices, it can be seen that \(wA_0 e = \frac{1}{M} \varphi \otimes \varphi (\sum_{i=1}^{M-1} \varphi \otimes \varphi) e = \frac{1}{M} \sum_{i=1}^{M-1} (\varphi \otimes \varphi) e
\) (16)}
Introducing the rate matrix \( R \) as the minimal non-negative solution of the non-linear matrix equation

\[
A_0 + RA_1 + R^2 A_2 = 0, \tag{21}
\]

can be proved (Neuts [7]) that \( \pi_n \) satisfies the following.

\[
\pi_n = \pi_n R^{-1} \quad \text{for} \quad n \geq 2. \tag{22}
\]

Using (18), \( \pi_0 \) satisfies

\[
\pi_0 = \pi_0 B_2 (B_1)^{-1}. \tag{23}
\]

So using (19) and (23) and (22) the vector \( \pi_1 \) can be calculated up to multiplicative constant since \( \pi_1 \) satisfies the equation \( \pi_1 B_2 (B_1)^{-1} \beta_0 + A_1 R A_2 \pi_0 = 0 \).

Using (17) and (23) it can be seen that

\[
\pi_1 [B_2 (B_1)^{-1}] e = \pi_0. \tag{25}
\]

Replacing the first column of the matrix multiplier of \( \pi_1 \) in equation (24), by the column vector multiplier of \( \pi_1 \) in (25), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is \( \pi_1 \) and this goes along with (22) and (23) all the stationary probabilities of the system.

The matrix \( R \) is iterated starting with \( R(0) = 0 \) and finding \( R(n+1) = A_0 A_1^{-1} R(n) A_2 A_1^{-1} \).

The iteration may be terminated to get a solution of \( R \) at a norm level where

\[
||R(n+1) - R(n)|| \leq \varepsilon.
\]

### 2.3 Performance Measures of the System

(i) The probability of the queue length \( S = r \), \( P(S=r) \) can be seen.

(ii) Let \( n \geq 0 \) and \( k \) for \( 0 \leq k \leq M-1 \) be non-negative integers such that \( r = n + M \cdot k \).

(iii) The expected queue level \( E(S) \), can be calculated.

(iv) Variance of \( S \) can be derived.

(v) Introducing the rate matrix \( R \) as the minimal non-negative solution of the non-linear matrix equation

\[
A_0 + RA_1 + R^2 A_2 = 0, \tag{21}
\]

can be proved (Neuts [7]) that \( \pi_n \) satisfies the following.

\[
\pi_n = \pi_n R^{-1} \quad \text{for} \quad n \geq 2. \tag{22}
\]

Using (18), \( \pi_0 \) satisfies

\[
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So using (19) and (23) and (22) the vector \( \pi_1 \) can be calculated up to multiplicative constant since \( \pi_1 \) satisfies the equation \( \pi_1 B_2 (B_1)^{-1} \beta_0 + A_1 R A_2 \pi_0 = 0 \).

Using (17) and (23) it can be seen that

\[
\pi_1 [B_2 (B_1)^{-1}] e = \pi_0. \tag{25}
\]

Replacing the first column of the matrix multiplier of \( \pi_1 \) in equation (24), by the column vector multiplier of \( \pi_1 \) in (25), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is \( \pi_1 \) and this goes along with (22) and (23) all the stationary probabilities of the system.

The matrix \( R \) is iterated starting with \( R(0) = 0 \) and finding

\[
R(n+1) = A_0 A_1^{-1} R(n) A_2 A_1^{-1}, \quad n \geq 0.
\]

The iteration may be terminated to get a solution of \( R \) at a norm level where

\[
||R(n+1) - R(n)|| < \varepsilon.
\]

### 3. MODEL (B) MAXIMUM ARRIVAL SIZE \( M < \) MAXIMUM SERVICE SIZE \( N \)

The dual case of Model (A), namely the case, \( M < N \) is treated here. (When \( M = N \) both models are applicable and one can use any one of them.) The assumption (v) of Model (A) is changed and all its other assumptions are retained.

#### 3.1 Assumption

(v). The maximum arrival size \( M = \max_{i \leq g \leq k} M_i \) is less than the maximum service size \( N = \max_{i \leq g \leq k} N_j \).

#### 3.2 Analysis

Since this model is dual, the analysis is same as that of Model (A). The differences are noted below. The state space of the chain is as follows presented in a similar way.

\( X(t) = (0, i) : \{0 \leq i \leq k\} \cup \{0, k, i, j\} : \{0 \leq k \leq N-1, 1 \leq i \leq k\} \cup (n, k, i, j) : \{0 \leq k \leq N-1, 1 \leq i \leq k, 1 \leq j \leq N\} \)

The chain is in the state (0, i) when the number of customers in the queue is 0, and the arrival phase is i for 1 \( \leq i \leq k \). The chain is in the state (0, k, i, j) when the number of customers in the queue is n and the service phase is j for 1 \( \leq j \leq k \).

The states of the chain are identified with (n, k) where r on division by N gives n as the quotient and k as the remainder. The infinitesimal generator \( Q_{ij} \) of the model has the same block partitioned structure given in (4) for Model (A) but the inner matrices are of different

\[
\begin{bmatrix}
B_1 & B_0 & 0 & \ldots & \ldots \\
B_2 & A_1 & A_0 & \ldots & \ldots \\
0 & A_2 & A_1 & A_0 & \ldots \\
0 & 0 & A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots & \vdots & \ldots
\end{bmatrix}
\tag{28}
\]

In (28) the states of the matrices are listed lexicographically as

\( 0, 1, 2, 3, \ldots, n \). For partition purpose the zero states in the first two sets of (27) are combined. All states are listed as follows. The vector \( 0 \) is of type 1 \( x \) \( k_1 \) (\( k_1, k_2, k_2(N-1) \)) and the vector \( n \) is of type 1 \( x \) \( k_1, k_2, N \). They are as follows.

\( \eta = (0, 0, 1, 2, 3, \ldots, \{k_1\}, \{0, 1, 1, 1\}, \{0, 1, 1, 2, \ldots, \{0, N-1, 1\}\}) \)

The matrices \( B_1 \) and \( A_1 \) are of orders \( k_1 + k_2 k_2(N-1) \) and \( k_1 k_2 k_2 \) respectively. They have negative diagonal elements and their off diagonal elements are non-negative. The matrices \( A_0 \) and \( A_2 \) have nonnegative elements and are square matrices of order \( k_1 k_2 k_2 \). The matrices \( B_0 \) and \( B_2 \) have non-negative elements and are of type \([k_1 + k_1 k_2 k_2(N)] \times [k_1 + k_2 k_2(N)] \) and \([k_1 + k_1 k_2 k_2(N)] \times [k_1 + k_2 k_2(N)] \) using Model (A) for definitions of \( A_1 \) and \( A_1' \), for \( 1 \leq j \leq M \), and \( U_{ij}, V_{ij} \) for \( 1 \leq j \leq N \), and U and letting \( Q_{ij} = T \), the partitioning matrices are defined as follows. The matrix \( B_1 \) is same as that of \( A_0 \) with first zero block row is of order \( k_1 \) \( k_2, k_2 \) \( N \). The matrix \( B_2 \) is same as that of \( A_2 \) except the first column block which is of type \( k_1 \) \( k_2, k_2 \) and is \( (0, 0, 0, \ldots, 0)' \) where \( U_{ij} = k_1, S_{ij} = N \).
\[
A' = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
A_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
A_{M-1} & A_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\
& & & & & & & \\
A_2 & A_3 & \cdots & A_M & 0 & 0 & \cdots & 0 \\
A_1 & A_2 & \cdots & A_{M-1} & A_M & 0 & \cdots & 0 \\
\end{bmatrix}
\]

(29) \[
A'_2 = \begin{bmatrix}
U_N & U_{N-1} & U_{N-2} & \cdots & U_3 & U_2 & U_1 \\
0 & U_N & U_{N-1} & \cdots & U_5 & U_3 & U_2 \\
0 & 0 & U_N & \cdots & U_5 & U_4 & U_3 \\
0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} \\
0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & U_N \\
\end{bmatrix}
\]

(30) \[
A'_1 = \begin{bmatrix}
Q'_1 & A_1 & A_2 & \cdots & A_M & 0 & 0 & \cdots & 0 \\
U_1 & Q'_1 & A_1 & \cdots & A_{M-1} & A_M & 0 & \cdots & 0 \\
U_2 & U_1 & Q'_1 & A_1 & \cdots & A_M & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & Q'_1 & A_1 & A_2 & \cdots & A_{M-1} & A_M \\
U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q'_1 & A_1 & \cdots & A_M & 0 \\
U_{N-M+1} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q'_1 & \cdots & A_{M-2} & A_{M-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
U_{N-2} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-4} & \cdots & Q'_1 & A_1 \\
U_{N-1} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & U_1 & Q'_1 \\
\end{bmatrix}
\]

(31) \[
B'_1 = \begin{bmatrix}
T & A'_1 & A'_2 & \cdots & A'_M & 0 & 0 & \cdots & 0 \\
U & Q'_1 & A_1 & \cdots & A_{M-1} & A_M & 0 & \cdots & 0 \\
V'_1 & U_1 & Q'_1 & A_1 & \cdots & A_{M-2} & A_{M-1} & A_M & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
V_{N-M-2} & V_{N-M-3} & \cdots & Q'_1 & A_1 & A_2 & \cdots & A_{M-1} & A_M \\
V_{N-M-1} & V_{N-M} & V_{N-M-1} & \cdots & U_1 & Q'_1 & A_1 & \cdots & A_{M-2} & A_{M-1} \\
V_{N-M} & V_{N-M-1} & V_{N-M} & \cdots & U_2 & U_1 & Q'_1 & \cdots & A_{M-3} & A_{M-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
V_{N-3} & V_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-4} & \cdots & Q'_1 & A_1 \\
V_{N-2} & V_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & U_1 & Q'_1 \\
\end{bmatrix}
\]

(32) \[
Q''_B = \begin{bmatrix}
Q'_1 + U_N & A_1 + U_{N-1} & \cdots & A_{M-1} + U_{N-M+1} & A_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 \\
U_1 & Q'_1 & A_1 & \cdots & A_{M-2} + U_{N-M+2} & A_{M-1} + U_{N-M+1} & A_M + U_{N-M} & \cdots & U_3 & U_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_{N-M-2} & U_{N-M-3} & \cdots & Q'_1 & A_1 & A_2 + U_{N-2} & A_{M-1} + U_{N-1} & A_M + U_N & U_{N-M-1} \\
U_{N-M-1} & U_{N-M} & U_{N-M-1} & \cdots & U_1 & Q'_1 & A_1 & A_2 + U_{N-2} & A_{M-1} + U_{N-1} & A_M + U_N \\
A_M + U_N & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q'_1 & A_1 & A_2 + U_{N-2} & A_{M-1} + U_{N-1} \\
A_1 + U_{N-1} & A_2 + U_{N-2} & \cdots & A_M + U_N & U_{N-M} & U_{N-M-1} & \cdots & U_2 & Q'_1 & A_1 & A_2 + U_{N-2} & A_{M-1} + U_{N-1} \\
\end{bmatrix}
\]

(33) \[
T = \begin{bmatrix}
-3 & 1 & 1 \\
1 & -4 & 1 \\
2 & 1 & -5 \\
\end{bmatrix}
\quad \text{and } \alpha = (3, 4, 3) \text{ and the service time}
\]

\[
\beta = (A, .6). \text{ Six examples are studied. The maximum arrival size and maximum service size are fixed as M=4 and N=2 in two examples. Two examples each treat the cases M=3 and M=2 and N=4. The order of the rate matrix R is 24 since it is the product of two matrices depending on M>N or N>M. The probabilities of bulk arrival sizes and bulk service sizes are varied in the examples. The bulk arrival probabilities of sizes 1, 2, 3 and 4 in the examples 1 and 3 are (.5, .2, .2, .1), (.5, .3, .2, 0) and (.5, .4, .1, 0) in arrival phases 1, 2, 3, respectively. In examples 2 and 4 they are (.5, .2, .2, 1), (.8, .2, 0, 0) and (.5, .4, .1, 0) in arrival phases 1, 2, 3, respectively. In example 5 they are (.5, .5, 0, 0), (.6, .4, 0, 0) and (.4, .6, 0, .6).
\]

4. NUMERICAL ILLUSTRATIONS

For numerical illustration it is considered that the arrival time PH distribution has representation
and bulk service has number of applications. The PH distributions include Exponential, Erlang, Hyper Exponential, and Coxian distributions as special cases and the PH distribution is also a best approximation for a general distribution. Further the PH/PH/1 queue is a most general distribution. Further the PH/PH/1 queue is a most general PH service phases 1 and 2. The iteration for the rate matrix R is performed for the same 20 number of times in all the six examples. When the arrival rates decrease, the probability of empty queue increases, the norm value of the convergence of the rate matrix for the same 20 iterations decreases, expected values of queue length and its variances also decrease. The situation is seen same for all the six models. The arrival rates, service rates, probabilities of empty queue, probabilities of queue length is 1; 2; 3; in between 0 and 3; in between 4 and 7; in between 8 and 11, in between 12 and 15; and greater than 15 are given in table 1. Twenty iterations are performed and the difference norms are presented in the table along with the expected queue lengths and the variances for the six examples. The decrease in the arrival rate decreases the expected queue length and variances. Figure 1 presents the variations seen in E(S) and π0e for various maximum bulk sizes graphically. In Figure 2 the several of queue lengths probabilities for the six cases are exhibited.

Table 1. Results obtained for six examples

<table>
<thead>
<tr>
<th></th>
<th>M=4, N=2</th>
<th>M=4, N=2</th>
<th>M=N=4</th>
<th>M=N=4</th>
<th>M=2, N=4</th>
<th>M=2, N=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arrival rate</td>
<td>0.676819923</td>
<td>0.676819923</td>
<td>0.595402299</td>
<td>0.595402299</td>
<td>0.585057471</td>
<td>0.545785441</td>
</tr>
<tr>
<td>Service rate</td>
<td>1.125000000</td>
<td>1.125000000</td>
<td>1.100000000</td>
<td>1.100000000</td>
<td>1.200000000</td>
<td>1.200000000</td>
</tr>
<tr>
<td>P(Q=0)</td>
<td>0.343363192</td>
<td>0.403076054</td>
<td>0.325905977</td>
<td>0.386498364</td>
<td>0.438311803</td>
<td>0.461638676</td>
</tr>
<tr>
<td>P(Q=1)</td>
<td>0.122523701</td>
<td>0.150694353</td>
<td>0.118911487</td>
<td>0.148288409</td>
<td>0.138481187</td>
<td>0.158525026</td>
</tr>
<tr>
<td>P(Q=2)</td>
<td>0.107999639</td>
<td>0.118735406</td>
<td>0.106399987</td>
<td>0.118346458</td>
<td>0.146464243</td>
<td>0.140139153</td>
</tr>
<tr>
<td>P(Q=3)</td>
<td>0.092853695</td>
<td>0.088989566</td>
<td>0.092338243</td>
<td>0.090120570</td>
<td>0.081938119</td>
<td>0.080552227</td>
</tr>
<tr>
<td>π0e</td>
<td>0.666740226</td>
<td>0.761495379</td>
<td>0.643555694</td>
<td>0.743253800</td>
<td>0.805195352</td>
<td>0.840855082</td>
</tr>
<tr>
<td>π1e</td>
<td>0.204961558</td>
<td>0.172011845</td>
<td>0.210841275</td>
<td>0.180196702</td>
<td>0.151642922</td>
<td>0.130042415</td>
</tr>
<tr>
<td>π2e</td>
<td>0.078850403</td>
<td>0.086758557</td>
<td>0.086060837</td>
<td>0.053624185</td>
<td>0.033508869</td>
<td>0.023744555</td>
</tr>
<tr>
<td>π3e</td>
<td>0.030386571</td>
<td>0.013407105</td>
<td>0.035189014</td>
<td>0.016057352</td>
<td>0.007492223</td>
<td>0.004370976</td>
</tr>
<tr>
<td>P(Q&gt;15)</td>
<td>0.019061241</td>
<td>0.005219917</td>
<td>0.024353180</td>
<td>0.006876962</td>
<td>0.002160634</td>
<td>0.000969673</td>
</tr>
<tr>
<td>Norm</td>
<td>0.000016888</td>
<td>0.000005259</td>
<td>0.000028460</td>
<td>0.00001239</td>
<td>0.000001239</td>
<td>0.000000000</td>
</tr>
<tr>
<td>E(S)</td>
<td>3.187053542</td>
<td>2.242943673</td>
<td>3.456788020</td>
<td>2.409022886</td>
<td>1.880878826</td>
<td>1.617146983</td>
</tr>
<tr>
<td>VAR(S)</td>
<td>16.8847935</td>
<td>9.300504294</td>
<td>10.4091890</td>
<td>19.2781613</td>
<td>10.4091890</td>
<td>5.216230045</td>
</tr>
</tbody>
</table>

5. CONCLUSION

Two PH/PH/1 bulk arrivals and bulk service queues have been treated by identifying the maximum of the arrival and service sizes and grouping the customers as members of blocks of such maximum sizes. Matrix geometric results have been obtained by partitioning the infinitesimal generator by grouping of customers and PH phases together. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard forms. Numerical results for varying bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. The variances also decrease. The PH/PH/1 queue with bulk arrival

![Figure 1. The values E(S) and π₀e.](image1)

![Figure 2. The probabilities of queue lengths.](image2)
Further studies with block circulant basic generator system may produce interesting and useful results in inventory theory and finite storage models like dam theory. It is also noticed here that once the maximum arrival or service size increases, the order of the rate matrix increases proportionally. However the matrix geometric structure is retained and rates of convergence is not much affected. Randomly varying environments causing changes in the sizes of the PH phases may produce further results if studied with suitable partition techniques.

6. REFERENCES