Extended Jacobian Elliptic Function Expansion Method and its Applications for Solving some Nonlinear Evolution Equations in Mathematical Physics

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ABSTRACT
Extended Jacobian elliptic function expansion method is employed to find the exact traveling wave solutions involving parameters for nonlinear evolution equations. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. It is shown that extended Jacobian elliptic function expansion method provides an effective and a more powerful mathematical tool for solving nonlinear evolution equations in mathematical physics. Comparison between our results and the well-known results will be presented.

Keywords
Extended Jacobian elliptic function expansion method ; (2+l)-Dimensional soliton breaking equation; (3+l)-Dimensional Kadomstev-Petviash-vili; Tarveling wave solutions; Solitary wave solutions.

1. INTRODUCTION
The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Exact solutions for these equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, Optics, Plasma physics arid so on. Recently many new approaches for finding these solutions have been proposed, for example, tanh - sech method [2]-[4], extended tanh - method [5]-[7], sine - cosine method [8]-[10], homogeneous balance method [11, 12], F-expansion method [13]-[15], exp-function method [16, 17], trigonometric function series method [18], $(\frac{G'}{G})$-expansion method [19]-[22], Jacobi elliptic function method [23]-[26],and so on.

The objective of this article is to investigate more applications than obtained in [23]-[26] to justify and demonstrate the advantages of extended Jacobian elliptic function expansion method. Here, we apply this method to (2+l)-dimensional soliton breaking equation [27] and (3+l)-dimensional Kadomstev-Petviash-vili.

2. DESCRIPTION OF METHOD
Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx:} ....) = 0, \tag{2.1}$$

where F is polynomial in $u(x,t\}$ and its partial derivatives in which the highest oreder derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [23] -[26]

Step l. Using the transformation

$$u = u(\xi), \quad \xi = x - ct, \tag{2.2}$$

where k and c are the wave number and wave speed, to reduce Eq.(2.1) to the following

ODE:

$$P(u, u', u''', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$and its total derivatives, while ' $= \frac{d}{d\xi}$.

Step2. Making good use of extended Jacobian elliptic functions, we assume that (2.3) has the

solutions in these forms:

$$u(\xi) = a_0 + \sum_{j=1}^{N} f_i^{j-1}(\xi)\left[a_j f_j(\xi) + b_j g_j(\xi)\right], \ i = 1,2,3, ... \tag{2.4}$$

With

$$\begin{aligned} f_1(\xi) &= sn\xi, & g_1(\xi) &= cn\xi, \\ f_2(\xi) &= sn\xi, & g_2(\xi) &= dn\xi, \\ f_3(\xi) &= ns\xi, & g_3(\xi) &= cs\xi, \\ f_4(\xi) &= ns\xi, & g_4(\xi) &= ds\xi, \\ f_5(\xi) &= sc\xi, & g_5(\xi) &= nc\xi, \\ f_6(\xi) &= sd\xi, & g_6(\xi) &= nd\xi, \end{aligned} \tag{2.5}$$

where $sn\xi$, $cn\xi$, $dn\xi$, are the Jacobian elliptic sine function, The Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaisher's symbols and are generated by these three kinds of functions, namely

$$ns\xi = \frac{1}{sn\xi}, nc\xi = \frac{1}{cn\xi}, nd\xi = \frac{1}{dn\xi}, sc\xi = \frac{cn\xi}{sn\xi},$$

$$cs\xi = \frac{sn\xi}{cn\xi}, ds\xi = \frac{dn\xi}{sn\xi}, sd\xi = \frac{sn\xi}{dn\xi},$$

that have the relations

$$sn^2\xi + cn^2\xi = I, dn^2\xi + m^2sn^2\xi = 1, ns^2\xi = 1 + cs^2\xi, \tag{2.7}$$

$$ns^2\xi = m^2 + ds^2\xi, sc^2\xi + 1 = nc^2\xi, m^2sd^2 + 1 = nd^2\xi$$

with the modulus m $(0 < m < 1)$. In addition we know that

$$\frac{d}{d\xi}sn\xi = cn\xi dn\xi, \frac{d}{d\xi}cn\xi = -sn\xi dn\xi, \frac{d}{d\xi}dn\xi = -m^2 sn\xi cn\xi \tag{2.8}$$

The derivatives of other Jacobian elliptic functions are obtained by using Eq.(2.8). To balance the highest order linear term with nonlinear term we define the degree of u as $D[u] = n$ which gives rise to the degrees of other expressions as

$$D\left[\frac{d^q u}{d\xi^q}\right] = n + q, \quad D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^8\right] = np + s(n + q). \tag{2.9}$$

According the rules, we can balance the highest order linear term and nonlinear term in Eq.(2.3) so that *n* in Eq.(2.4) can be determined.

In addition we see that when $m \Rightarrow 1$, snξ, cnξ, and $dn\xi$ degenerate as $tanh\xi$, $sech\xi$, $cosech\xi$, respectively, while when therefore Eq.(2.5) degenerate as the following forms

$$u(\xi) = a_0 + \sum_{j=1}^{N} tanh^{j-1}(\xi)\left[a_j\, tanh(\xi) + b_j\, sech(\xi)\right], \tag{2.10}$$

$$u(\xi) = a_0 + \sum_{j=1}^{N} coth^{j-1}(\xi)\left[a_j\, coth(\xi) + b_j\, coth(\xi)\right], \tag{2.11}$$

$$u(\xi) = a_0 + \sum_{j=1}^{N} tan^{j-1}(\xi)\left[a_j\, tan(\xi) + b_j\, sec(\xi)\right], \tag{2.12}$$

$$u(\xi) = a_0 + \sum_{j=1}^{N} cot^{j-1}(\xi)\left[a_j\, cot(\xi) + b_j\, csc(\xi)\right], \tag{2.13}$$

Therefore the extended Jacobian elliptic function expansion method is more general than *sine-cosine* method, the ton-function method and Jacobian elliptic function expansion method.

3. APPLICATION

Here, we will apply extended Jacobian elliptic function expansion method described in sec.2 to find the exact traveling wave solutions and then the solitary wave solutions for the following nonlinear systems of evolution evolution equations.

3.1-Example 1: The (2+l)-dimensional breaking soliton equations

Let us consider the (2+l)-dimensional breaking soliton equations [27]:

$$\begin{cases} u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0, \\ u_y = v_x, \end{cases}$$

where *a* is known constant. Eqs.(3.1) describes the (2+l)-dimensional interaction of a Riemann wave propagating along the y-axis with along wave along the x-axis. In the past years, many authors have studied Eqs.(3.1). For instance, Zhang has successfully extended the generalized auxiliary equation method of the (2+l)-dimensional breaking soliton equations in [28]. Some soliton-like solutions were obtained by the generalized expansion of Riccati equation in [29]. Recently, a class of periodic wave solutions were obtained by the mapping method in [30]. Two classes of new exact solutions were obtained by the singular manifold method in [31]. Using the wave variable $\xi = x + y - ct$ and proceeding as before we find

$$\begin{cases} -cu' + \alpha u''' + 4\alpha u v' + 4\alpha u' v = 0, \\ u' = v', \end{cases}$$

Integrating the second equation in the system and neglecting constant of integration we find

$$u = v. \tag{3.3}$$

Substituting (3.3) into the first equation of the system and integration we find

$$-cu + 4\alpha u^2 + au'' = 0. \tag{3.4}$$

Balancing u^2 and u'' in Eq.(3.4) yields, $2N = N + 2 \Rightarrow N = 2$. Consequently, we get the formal solution

$$u(\xi) = a_0 + a_1 sn + b_1 cn + a_2 sn^2 + b_2 sncn, \tag{3.5}$$

where a_o, a_1, a_2 are constants to be determined, such that $a_2 \neq 0$ or $b_2 \neq 0$. It is easy to see that

$$u' = a_1 cn\, dn — b_1 sn\, dn + 2\, dn a_2 sn\, cn — 2\, dn b_2 sn^2 + dn b_2, \tag{3.6}$$

$$u'' = m^2 sn a_1 + 2a_1 sn^3 m^2 + 2m^2 sn^2 cn b_1 - 4a_2 m^2 sn^2 + 6a_2 sn^4 m^2 + 6m^2 sn^3 cn b_2 - m^2 sncn b_2 - a_1 sn - b_1 cn + 2a_2 - 4a_2 sn^2 - 4b_2 sncn. \tag{3.7}$$

Substituting (3.5) arid (3.7) into Eq.(3.4) arid equating all the coefficients of sn^4, sn^3cn, sn^3, sn^2cn, sn^2, *sncn, sn, en, sn°* to zero, we deduce respectively

$$4\, \alpha\, (a_2^2 - b_2^2) + 6\alpha\alpha_2 m^2 = 0, \tag{3.8}$$

$$8\alpha a_2 b_2 + 6\alpha m^2 b_2 = 0, \tag{3.9}$$

$$4\, a\, (-2\, b_1 b_2 + 2\, a_1\, a_2) + 2\, \alpha\, a_1 m^2 = 0, \tag{3.10}$$

$$4\, a\, (2a_1 b_2 + 2b_1 a_2) + 2\, \alpha\, b_1 m^2 = 0, \tag{3.11}$$

$$-ca_2 + 4\, \alpha\, (a_1^2 – b_1^2 + 2\, a_0 a2 + b_2^2) + \alpha\, (-4\, a_2 m^2 - 4\, a_2) = 0, \tag{3.12}$$

$$-cb_2 + 4\, \alpha\, (2\, a_1 b_1 + 2\, a_0 b_2) + a\, (-m^2 b_2 - 4\, b_2) = 0, \tag{3.13}$$

$$-ca_1 + 4a\, (2b_1 b_2 + 2a_0 a_1) + \alpha\, (-m^2 a_1 – a_1) = 0, \tag{3.14}$$

$$-cb_1 + 8\, \alpha\, a_0 b_1 - ab_1 = 0, \tag{3.15}$$

$$-ca_0 + 4\, \alpha\, (a_0^2 + b_1^2) + 2\, \alpha\, a_2 = 0. \tag{3.16}$$

From Eqs.(3.8)-(3.16), we have the following results:

$$c = 8\alpha\left(\frac{1}{2}m^2 + \frac{1}{2} - \frac{1}{2}\sqrt{m^4 - m^2 + 1}\right) - 4\alpha m^2 - 4\alpha,$$

$$a_0 = \frac{1}{2}m^2 + \frac{1}{2} - \frac{1}{2}\sqrt{m^4 - m^2 + 1}, a_1 = b_1 = b_2 = 0, a_2 = \frac{-3}{2}m^2.$$

So that the exact solution of Eq.(3.4)

$$u(\xi) = \tfrac{1}{2}m^2 + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{m^4 - m^2 + 1} - \tfrac{3}{2}m^2 sn^2, \tag{3.17}$$

now, if m → 1 we can obtain the hyperbolic solution:

$$u(\xi) = \tfrac{1}{2} - \tfrac{3}{2}tanh^2(\xi), \tag{3.18}$$

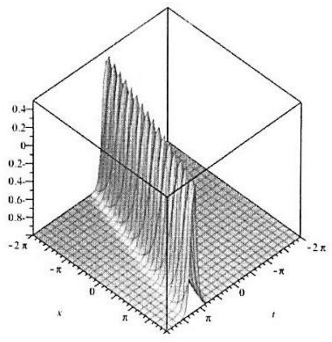

Figure 1: solution of Eq.(3.18)

3.2 Example 2: The (3+l)-dimensional KP equation

We next consider the (3+l)-dimensional KP equation

$$u_{xt} + 6u^2_x + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{3.19}$$

Xie et al. [32] obtained non-traveling wave solutions by the improved tanh function method, in which they introduced a generalized Riccati equation and gained its 27 new solutions. In this paper, we will construct new non-traveling wave solution of Eq.(2.1). As a result, new non-traveling wave solutions including soliton-like solutions and periodic solutions of Eq.(2.1) are obtained. A generalized variable-coefficient algebraic method with computerized symbolic computation is developed to deal with (3+l)-dimensional KP equation with variable coefficients in[33]. Cheri et al. [34]study (3+l)-dimensional KP equation by using the new generalized transformation in homogeneous balance method. Using the wave variable $\xi = x + y + z — ct$, the Eq.(3.19) is carried to an ODE of the form

$$-(c + 2)\, u'' + 6\,(u')^2 + 6uu'' - u''' = 0. \tag{3.20}$$

Integrating twice and setting the constants of integration to zero, we obtain

$$-(c + 2)\, u + 3u^2 - u'' = 0. \tag{3.21}$$

Balancing u'' and u^2 in Eq.(3.21) yields, $N + 2 = 2N \quad N = 2$. Consequently, we get the formal solution (3.5).

Substituting (3.5)-(3.7) into Eq.(3.21) and equating the coefficients of sn^4, sn^3cn, sn^3, sn^2cn, sn^2, $sncn$, sn, en, $sn°$ to zero, we respectively obtain

$$3a_2^2 - 3b_2^2 - 6a_2m^2 = 0, \tag{3.22}$$

$$6a_2b_2 – 6m^2b_2 = 0, \tag{3.23}$$

$$-6\, b_1b_2 + 6\, a_1a_2 - 2\, m^2a_1 = 0, \tag{3.24}$$

$$6\, a_1b_2 + 6\, b_1a_2 - 2\, b_1m^2 = 0, \tag{3.25}$$

$$-(c + 2)\, a_2 + 3\, a_1^2 - 3\, b_1^2 + 6\, a_0a_2 + 3\, b_2^2 + 4\, a_2m^2 + 4\, a_2 = 0, \tag{3.26}$$

$$-(c + 2)\, b_2 + 6\, a_1b_1 + 6\, a_0b_2 + m^2b_2 + 4\, b_2 = 0, \tag{3.27}$$

$$-(c + 2)\, a_1 + 6\, b_1b_2 + 6\, a_0a_1 + m^2\, a_1 + a_1 = 0, \tag{3.28}$$

$$-(c + 2)b_1 + 6a_0b_1 + b_1 = 0, \tag{3.29}$$

$$-(c + 2)\, a_0 + 4\, \alpha\, (a_0^2 + b_1^2) - 2\, a_2 = 0. \tag{3.30}$$

From Eqs.(3.22)-(3.30), we have the following results:

$$c = 2 + 6\frac{m^2 + 1 + \sqrt{m^4 - 4m^2 + 1 + 4\alpha m^2}}{-3 + 2\alpha} + 4m^2,$$

$$a_0 = \frac{m^2 + 1 + \sqrt{m^4 - 4m^2 + 1 + 4\alpha m^2}}{-3 + 2\alpha}, a_1 = b_1 = b_2 = 0, a_2 = 2m^2$$

So that the exact solution of Eq.(3.21)

$$u(\xi)=\frac{m^2+1+\sqrt{m^4-4m^2+1+4\alpha m^2}}{-3+2\alpha} + 2m^2sn^2, \tag{3.31}$$

now, if m —> 1 we can obtain the hyperbolic solution:

$$u(\xi)=\frac{2+\sqrt{-4+4\alpha}}{-3+2\alpha} + 2, tnah^2(\xi), \tag{3.32}$$

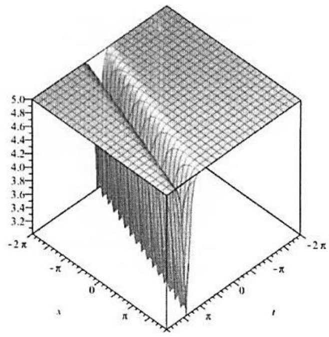

Figure 2: solution of Eq.(3.32)

4. CONCLUSION

Extended Jacobian elliptic function expansion method has been successfully used to find the exact traveling wave solutions of nonlinear evolution equations. As an application, the traveling wave solutions for (2+l)-dimensional soliton breaking equation and (3+l)-dimensional Kadomstev-Petviash-vili which have been constructed using the modified simple equation method. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of (2+l)-dimensional soliton breaking equation and (3+l)-dimensional Kadomstev-Petviash-viliare are new and different from those obtained in [35], [36]. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations.

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