Soft Bitopological Spaces

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ABSTRACT

In this paper, introduce and study the concept of soft bitopological spaces which are defined over an initial universe with a fixed set of parameters. Also introduce and investigate some new separation axioms called pairwise soft $T_0$, pairwise soft $T_1$ and pairwise soft $T_2$ spaces and study some of their basic properties in soft bitopological spaces.

1. INTRODUCTION

In the year 1999, Russian researcher Molodtsov[6], initiated the concept of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In 2003, Maji, Biswas and Roy[5], studied the theory of soft sets initiated by Molodtsov. They defined equality of two soft sets, subset and absolute soft set with examples. Soft binary operations like AND, OR and also the operations of union and intersection were also defined. In 2005, D. Chen[2], presented a new definition of soft set by

$$F(A) = \{ (\epsilon, \alpha) \mid \alpha \in A \}$$

In 1963, J. C. Kelly[4], first initiated the concept of bitopological spaces. He defined a bitopological space $(X, \tau_1, \tau_2)$ to be a set $X$ equipped with two topologies $\tau_1$ and $\tau_2$ on $X$ and initiated the systematic study of bitopological space. Later work done by C. W. Patty[8], I. L. Reilly[9] and others. Reilly discussed separation axioms properties in bitopological spaces.

The following definitions which are prerequisites for present study.

1.1 Definition[6]: Let $U$ be an initial universe and $E$ be a set of parameters. Let $P(U)$ denote the power set of $U$ and $A$ be a non-empty subset of $E$. A pair $(F,A)$ is called a soft set over $U$, where $F$ is a mapping given by $F: A \rightarrow P(U)$.

In other words, a soft set over $U$ is a parametrized family of subsets of the universe $U$. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of $\epsilon$-approximate elements of the soft set $(F,A)$. Clearly, a soft set is not a set.

1.2 Definition[5]: The complement of a soft set $(F,A)$ is denoted by $(F,A)^c$ and is defined by $(F,A)^c = (F^c, \neg A)$, where $F^c : \neg A \rightarrow P(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$ for all $\alpha \in \neg A$.

Let us call $F^c$ to be the soft complement function of $F$. Clearly $(F^c)^c$ is the same as $F$ and $((F,A)^c)^c = (F,A)$.

1.3 Definition[5]: A soft set $(F,A)$ over $U$ is said to be a NULL soft set denoted by $\emptyset$ if for all $\epsilon \in A$, $F(\epsilon) = \emptyset$ (null set).

1.4 Definition[5]: The union of two soft sets $(F,A)$ and $(G,B)$ over the common universe $U$ is the soft set $(H,C)$, where $C = A \cup B$ and for all $e \in E$,

$$H(\epsilon) = \begin{cases} F(\epsilon) & \text{if } \epsilon \in A - B \\ G(\epsilon) & \text{if } \epsilon \in B - A \\ F(\epsilon) \cup G(\epsilon) & \text{if } \epsilon \in A \cap B \end{cases}$$

and is written as $(F,A) \cup (G,B) = (H,C)$.

1.5 Definition[3]: The intersection $(H,C)$ of two soft sets $(F,A)$ and $(G,B)$ over a common universe $U$, denoted $(F,A) \cap (G,B)$, is defined as $C = A \cap B$, and $H(\epsilon) = F(\epsilon) \cap G(\epsilon)$ for all $e \in C$.

1.6 Definition[7]: Let $(F,E)$ be a soft set over $X$ and $x \in X$. We say that $x \in (F,E)$ as $x$ belongs to the soft set $(F,E)$ whenever $x \in F(\alpha)$ for all $\alpha \in E$.
Note that for any $x \in X$, $x \notin (F, E)$ if $x \notin F(\alpha)$ for some $\alpha \in E$.

1.7 Definition[7]: Let $Y$ be a non-empty subset of $X$, then $Y$ denotes the soft set $(Y, \tau)$ over $X$ for which $Y(\alpha) = Y$, for all $\alpha \in E$.

1.8 Definition[7]: Let $x \in X$, then $(x, E)$ denotes the soft set over $X$ for which $x(\alpha) = \{x\}$, for all $\alpha \in E$.

1.9 Definition[7]: Let $(F, E)$ be a soft set over $X$ and $Y$ be a non-empty subset of $X$. Then the sub soft set of $(F, E)$ over $Y$ denoted by $(Y, F, E)$, is defined as follows

$$Y F(\alpha) = Y \cap F(\alpha),$$

for all $\alpha \in E$. In other words $(Y, F, E) = Y \cap (F, E)$.

1.10 Definition[7]: The relative complement of a soft set $(F, A)$ is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ where $F^c : A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$ for all $\alpha \in A$.

1.11 Definition[7]: Let $\tau$ be the collection of soft sets over $X$, then $\tau$ is said to be a soft topology on $X$, if

1. $\phi, X$ belong to $\tau$
2. The union of any number of soft sets in $\tau$ belongs to $\tau$
3. The intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a soft topological space over $X$.

1.12 Definition[7]: Let $(X, \tau, E)$ be a soft topological space over $X$, then the members of $\tau$ are said to be soft open sets in $X$.

1.13 Definition[7]: Let $(X, \tau, E)$ be a soft topological space over $X$. A soft open set $(F, E)$ over $X$ is said to be a soft closed set in $X$, if its relative complement $(F, E)^c$ belongs to $\tau$.

1.14 Definition[7]: Let $(X, \tau, E)$ be a soft topological space over $X$ and $Y$ be a non-empty subset of $X$. Then $\tau_Y = \{ (Y, F, E) : (F, E) \in \tau \}$ is said to be the soft relative topology on $Y$ and $(Y, \tau_Y, E)$ is called a soft subspace of $(X, \tau, E)$.

1.15 Definition[7]: Let $(X, \tau, E)$ be a soft topological space over $X$ and $x, y \in X$ such that $x \neq y$.

i) If there exist soft open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E)$ and $y \notin (F, E)$ or $y \in (G, E)$ and $x \notin (G, E)$, then $(x, \tau, E)$ is called a soft $T_0$ space.

ii) If there exist soft open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E)$ and $y \notin (F, E)$ and $y \in (G, E)$ and $x \notin (G, E)$, then $(x, \tau, E)$ is called a soft $T_1$ space.

iii) If there exist soft open sets $(F, E)$ and $(G, E)$ such that $x \in (F, E), y \in (G, E)$ and $(F, E) \cap (G, E) = \phi$, then $(x, \tau, E)$ is called a soft $T_2$ space.

1.16 Definition[4]: Let $X$ be a non-empty set and $\tau_1$ and $\tau_2$ be two different topologies on $X$. Then $(X, \tau_1, \tau_2)$ is called a bitopological space.

### 2. SOFT BITOPOLITICAL SPACES

Let $X$ be an initial universe set and $E$ be the non-empty set of parameters.

2.1 Definition: Let $(X, \tau_1, E)$ and $(X, \tau_2, E)$ be the two different soft topologies on $X$. Then $(X, \tau_1, \tau_2, E)$ is called a soft bitopological space.

The two soft topologies $(X, \tau_1, E)$ and $(X, \tau_2, E)$ are independently satisfy the axioms of soft topology. The members of $\tau_1$ are called $\tau_1$ soft open sets and the complements of $\tau_1$ soft open sets are called $\tau_1$ soft closed sets.

Similarly, the members of $\tau_2$ are called $\tau_2$ soft open sets and the complements of $\tau_2$ soft open sets are called $\tau_2$ soft closed sets.

Throughout this paper $(X, \tau_1, \tau_2, E)$ denote soft bitopological space over $X$ on which no separation axioms are assumed unless explicitly stated.

2.2 Example: Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau_1 = \{\phi, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$ and $\tau_2 = \{\phi, X, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)\}$, where $(F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E), (G_7, E)$ are soft sets over $X$, defined as follows

$F_1(e_1) = \{h_3\}$ $F_1(e_2) = \{h_1\}$
$F_2(e_1) = \{h_2, h_3\}$ $F_2(e_2) = \{h_1, h_3\}$
$F_3(e_1) = \{h_1, h_2\}$ $F_3(e_2) = X$
$F_4(e_1) = \{h_1, h_2\}$ $F_4(e_2) = \{h_1, h_3\}$
$F_5(e_1) = \{h_2\}$ $F_5(e_2) = \{h_1, h_2\}$

$G_1(e_1) = \{h_1, h_2\}$ $G_1(e_2) = \{h_1, h_2\}$
$G_2(e_1) = \{h_2\}$ $G_2(e_2) = \{h_1, h_3\}$
$G_3(e_1) = \{h_1, h_2, h_3\}$ $G_3(e_2) = \{h_1\}$
$G_4(e_1) = \{h_1\}$ $G_4(e_2) = \{h_1\}$
$G_5(e_1) = \{h_1, h_2\}$ $G_5(e_2) = X$
$G_6(e_1) = X$ $G_6(e_2) = \{h_1, h_2\}$
$G_7(e_1) = \{h_2, h_3\}$ $G_7(e_2) = \{h_1, h_3\}$

Then $(X, \tau_1, \tau_2, E)$ is a soft bitopological space.

2.3 Example: Let $X$ be an initial universe set and $E$ be the non-empty set of parameters. Soft indiscreet topology $\tau_1 = \{\phi, X\}$ and soft discrete topology $\tau_2$ is the collection of all soft sets defined over $X$. Then $(X, \tau_1, \tau_2, E)$ is a soft bitopological space.

2.4 Definition: Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space over $X$ and $Y$ be a non-empty subset of $X$. Then $\tau_{1Y} = \{ (Y, F, E) : (F, E) \in \tau_1 \}$ and $\tau_{2Y} = \{ (Y, G, E) : (G, E) \in \tau_2 \}$ are said to be the relative topologies on $Y$. Then $(Y, \tau_{1Y}, \tau_{2Y}, E)$ is called a relative soft bitopological space of $(X, \tau_1, \tau_2, E)$.

2.5 Theorem: If $(X, \tau_1, \tau_2, E)$ is a soft bitopological space then $\tau = \tau_1 \cap \tau_2$ is a soft topological space over $X$.

Proof: 1). $\phi, X$ belong to $\tau_1 \cap \tau_2 = \tau$
2). Let $\{ (F_i, E) : i \in I \}$ be a family of soft sets in $\tau_1 \cap \tau_2 = \tau$.
Then $\{ (F_i, E) \in \tau_1 \}$ and $\{ (F_i, E) \in \tau_2 \}$ for all $i \in I$.
Therefore $\bigcup_{i \in I} (F_i, E) \in \tau_1$ and $\bigcup_{i \in I} (F_i, E) \in \tau_2$.
Thus $\bigcup_{i \in I} (F_i, E) \in \tau_1 \cap \tau_2 = \tau$.
3). Let $(F, E), (G, E) \in \tau_1 \cap \tau_2 = \tau$.
Then $(F, E), (G, E) \in \tau_1$ and $(F, E), (G, E) \in \tau_2$.
Since $(F, E) \cap (G, E) \in \tau_1$ and $(F, E) \cap (G, E) \in \tau_2$.
Therefore $(F, E) \cap (G, E) \in \tau_1 \cap \tau_2 = \tau$.
Thus $\tau_1 \cap \tau_2 = \tau$ defines a soft topology on $X$.

2.6 Remark: If $(X, \tau_1, \tau_2, E)$ is a soft bitopological space then $\tau_1 \cup \tau_2$ is not a soft topological space over $X$.

2.7 Example: Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tau_1 = \{\phi, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ and $\tau_2 = \{\phi, X, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E)\}$
where \((F_1, E_1), (F_2, E_2), (F_3, E_3), (F_4, E_4), (G_1, E_1), (G_2, E_2), \) are soft sets over \(X\), defined as follows

\[
F_1(e_1) = \{h_1, h_2\}, \quad F_2(e_2) = \{h_2, h_3\} \\
F_3(e_1) = \{h_1\}, \quad F_4(e_2) = \emptyset \\
G_1(e_1) = \{h_1, h_2\}, \quad G_2(e_2) = \{h_2, h_3\} \\
\]

Similarly \(G, E\) define a soft open set \(1 \cap \tau\) of \(X\) such that \(x \notin (F, E)\) and \(y \notin (F, E)\) or \(y \in (G, E)\) and \(x \notin (G, E)\). Now \(x \notin \tau\) and \(x \notin (E, F)\). Hence \(x \in Y \cap (F, E) = (Y, F, E)\) where \((F, E) \in \tau_1\). Consider \(y \notin (F, E)\) this means that \(y \in \alpha E\), then \(y \notin Y \cap F(\alpha)\) for some \(\alpha \in E\). Therefore \(y \notin Y \cap (F, E) = (Y, F, E)\). Therefore \(\tau_1\) is soft \(T_0\) space w.r.t. \(\tau_{2Y}\).

Similarly it can be prove that \(\tau_{2Y}\) is soft \(T_0\) space w.r.t \(\tau_{1Y}\), that is \(y \in (G, E)\) and \(x \notin (G, E)\) then \(x \notin (Y, E)\) and \(x \notin (Y, G, E)\).

Thus \((Y, \tau_{1Y}, \tau_{2Y}, E)\) is pairwise soft \(T_0\) space.

### 3.3 Theorem:
Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space over \(X\) and \(Y\) be a non-empty subset of \(X\). If \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_1\) space then \((Y, \tau_{1Y}, \tau_{2Y}, E)\) is pairwise soft \(T_1\) space.

**Proof:** Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space over \(X\) and \(x, y \in Y\) such that \(x \neq y\). If \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_0\) space, then there exist a \(\tau_1\) soft open set \((F, E)\) and a \(\tau_2\) soft open set \((G, E)\) such that \(x \in (F, E)\) and \(y \notin (F, E)\) or \(y \in (G, E)\) and \(x \notin (G, E)\). Now \(x \notin \tau_1\) and \(x \notin (E, F)\). Hence \(x \in Y \cap (F, E) = (Y, F, E)\) where \((F, E) \in \tau_1\). Therefore \(\phi G, E\) is pairwise soft \(T_0\) space.

### 3.4 Theorem:
Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space over \(X\) and \(x, y \in X\) such that \(x \neq y\). If \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_0\) space, then there exist \(\tau_1\) soft open set \((F, E)\) and \(\tau_2\) soft open set \((G, E)\) such that \(x \in (F, E)\) and \(y \notin (F, E)\) and \(y \in (G, E)\) and \(x \notin (G, E)\). Now \(x \notin Y\) and \(x \notin (F, E)\). Hence \(x \in Y \cap (F, E) = (Y, F, E)\) where \((F, E) \in \tau_1\). Consider \(y \notin (F, E)\) this means that \(\alpha \in E\), then \(y \notin Y \cap F(\alpha)\) for some \(\alpha \in E\). Therefore \(y \notin Y \cap (F, E) = (Y, F, E)\). Hence \(\tau_{1Y} \subset T_0\) space w.r.t. \(\tau_{2Y}\).

Similarly it can be prove that \(\tau_{2Y}\) is soft \(T_0\) space on \(x \notin \tau_{1Y}\), that is \(y \in (G, E)\) and \(x \notin (G, E)\) then \(x \notin (Y, E)\) and \(x \notin (Y, G, E)\).

Thus \((Y, \tau_{1Y}, \tau_{2Y}, E)\) is pairwise soft \(T_1\) space.

### 3.5 Theorem:
Every pairwise soft \(T_1\) space is pairwise soft \(T_0\) space.

**Proof:** Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space over \(X\) and \(x, y \in X\) such that \(x \neq y\). If \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_1\) space. That is \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_0\) space w.r.t \(\tau_{1Y}\) and \(\tau_{2Y}\) is soft \(T_1\) space w.r.t \(\tau_{1Y}\) and \(\tau_{2Y}\) is soft \(T_0\) space. If \(x \in (F, E)\) is a soft closed set in \(\tau_2\) then \((x, E)^c\) is soft open set in \(\tau_2\). Let \(x, y \in X\) such that \(x \neq y\). For each \(x \in (x, E)^c\) is a soft open set in \(\tau_2\) such that \(y \in (x, E)^c\) and \(x \notin (x, E)^c\). Similarly for every \(y \in X\), \((y, E)^c\) is a soft closed set in \(\tau_1\) then \((y, E)^c\) is soft open set in \(\tau_1\) such that \(x \in (y, E)^c\) and \(x \notin (y, E)^c\).

Thus \((X, \tau_1, \tau_2, E)\) is pairwise soft \(T_1\) space.
3.6 Remark: The converse of the Theorem 3.5 is not true.

3.7 Example: Let $X = \{h_1, h_2\}$, $E = \{e_1, e_2\}$ and $	au_1 = \{\phi, X, (F, E)\}$ and $	au_2 = \{\phi, X\}$, where $(F, E)$ is soft set over $X$, defined as follows

$$F(e_1) = \phi, F(e_2) = \{h_1\}$$

Then $(X, \tau_1, \tau_2, E)$ is a soft bitopological space. Also $(X, \tau_1, \tau_2, E)$ is a pairwise soft $T_0$ space but which is not a pairwise soft $T_1$ space, because $h_1, h_2 \in X$ with $h_1 \neq h_2$ and there do not exist $\tau_1$ soft open set $(F_1, E)$ and $\tau_2$ soft open set $(G_1, E)$ such that $h_1 \in (F_1, E)$ and $h_2 \notin (F_1, E)$ and $h_2 \in (G_1, E)$ and $h_1 \notin (G_1, E)$.

3.8 Theorem: Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space over $X$ and $Y$ be a non-empty subset of $X$. If $(X, \tau_1, \tau_2, E)$ is pairwise soft $T_2$ space then $(Y, \tau_1 Y, \tau_2 Y, E)$ is pairwise soft $T_2$ space.

Proof: Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space over $X$ and $x, y \in Y$ such that $x \neq y$. If $(X, \tau_1, \tau_2, E)$ is pairwise soft $T_2$ space, then there exist a $\tau_1$ soft open set $(F, E)$ and a $\tau_2$ soft open set $(G, E)$ such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E) \cap (G, E) = \phi$. So for each $\alpha \in E$, $x \in F(\alpha)$, $y \in G(\alpha)$ and $F(\alpha) \cap G(\alpha) = \phi$. This implies that $x \in Y \cap F(\alpha)$, $y \in Y \cap G(\alpha)$ and $F(\alpha) \cap G(\alpha) = \phi$. Hence $x \in (Y, F, E)$, $y \in (Y, G, E)$ and $(Y, F, E) \cap (Y, G, E) = \phi$. Therefore $(Y, F, E)$ is soft open set in $\tau_1 Y$, and $(Y, G, E)$ is soft open set in $\tau_2 Y$. Therefore $\tau_1 Y$ is soft $T_2$ space w. r. t. $\tau_2 Y$.

Similarly it can be prove that $\tau_2 Y$ is soft $T_2$ space w. r. t. $\tau_1 Y$. Thus $(Y, \tau_1 Y, \tau_2 Y, E)$ is pairwise soft $T_2$ space.

3.9 Theorem: Every pairwise soft $T_2$ space is pairwise soft $T_1$ space.

Proof: Let $(X, \tau_1, \tau_2, E)$ be a soft bitopological space over $X$ and $x, y \in X$ such that $x \neq y$. If $(X, \tau_1, \tau_2, E)$ is pairwise soft $T_2$ space. That is $(X, \tau_1, \tau_2, E)$ is pairwise soft $T_2$ space if $\tau_1$ is soft $T_2$ space w. r. t. $\tau_2$ and $\tau_2$ is soft $T_2$ space w. r. t. $\tau_1$. If $\tau_1$ is soft $T_2$ space w. r. t. $\tau_2$, then there exist a $\tau_1$ soft open set $(F, E)$ and a $\tau_2$ soft open set $(G, E)$ such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E) \cap (G, E) = \phi$. Obviously $x \in (F, E)$ and $y \notin (F, E)$ and $y \in (G, E)$ and $x \notin (G, E)$. Therefore $\tau_1$ is soft $T_1$ space w. r. t. $\tau_2$.

Similarly if $\tau_2$ is soft $T_2$ space w. r. t. $\tau_1$, then there exist a $\tau_2$-soft open set $(F, E)$ and a $\tau_1$-soft open set $(G, E)$ such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E) \cap (G, E) = \phi$. Obviously $x \in (F, E)$ and $y \notin (F, E)$ and $y \in (G, E)$ and $x \notin (G, E)$. Therefore $\tau_2$ is soft $T_1$ space w. r. t. $\tau_1$. Thus $(X, \tau_1, \tau_2, E)$ pairwise soft $T_1$ space.

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4. REFERENCES


