Line Graphs and Quasi-total Graphs

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ABSTRACT
The line graph, 1-quasi-total graph and 2-quasi-total graph are well-known. It is proved that if G is a graph consist of exactly m connected components G_i, 1 ≤ i ≤ m, then L(G) = L(G_1) ⊕ L(G_2) ⊕ ... ⊕ L(G_m) where L(G) denotes the line graph of G, and ‘⊕’ denotes the ring sum operation on graphs. The number of connected components in G is equal to the number of connected components in L(G) and also if G is a cycle of length n, then L(G) is also a cycle of length n. The concept of 1-quasi-total graph is introduced and obtained that Q_1(G) = G ⊕ L(G) where Q_1(G) denotes 1-quasi-total graph of a given graph G. It is also proved that for a 2-quasi-total graph of G, the two conditions (i) |E(G)|= 1; and (ii) Q_2(G) contains unique triangle are equivalent.

General Terms
Graph Theory, line graphs, ring sum operation on graphs.

Keywords
Line graph, quasi-total graph, connected component.

1. LINE GRAPHS
All graphs considered are finite and simple. For standard literature on graph theory we refer Bondy and Murty [1], Harary [2], Satyanarayana and Syam Prasad [7, 8].

We start this section with the following remark.

1.1 Remark: Let G be a graph with E(G) ≠ ∅, and L(G) its line graph.

(i) V(G) ⊇ V(L(G)) = ∅.

(ii) E(G) ⊇ E(L(G)) = ∅.

1.2 Lemma: Let G be a graph. Suppose G_1, G_2 are two connected components of a graph G. Write G = G_1 ⊕ G_2. Suppose e_1, e_2 ∈ G such that s = e_1e_2 ∈ E(L(G)). If e_1 ∈ E(G_1), then e_2 ∈ E(G_1) but not e_2 ∈ E(G_2) (in other words, if s_1 ∈ V(G_1) and s_2 ∈ V(G_2), then s_1 and s_2 cannot be adjacent in L(G)).

Proof: Suppose e_1 ∈ G_1. Since e_2 ∈ E(G) = E(G_1) ∪ E(G_2), either e_2 ∈ E(G_1) or e_2 ∈ E(G_2). Now to show that e_2 /∈ E(G_2). If possible, suppose that e_2 ∈ E(G_2).

Since s = e_1e_2 ∈ E(L(G)), we have that e_1 is adjacent to e_2. Then e_1 ∈ V_1 and e_2 ∈ V_2 for some V_1, V_2, V_3 ∈ V(G).

Since e_1 ∈ E(G_1) and e_2 ∈ E(G_2), we have that V_1, V_2, V_3 ∈ V(G). So V_2 ∈ E(L(G)) ∩ E(G).

Take x, y ∈ G_1 ∪ G_2. If x, y ∈ G_1 (or G_2), then since G_1 (or G_2) is a connected component, we have that there is a path from x to y. Suppose x ∈ G_1 and y ∈ G_2.

Since x, v_2 ∈ G_1, there is a path from x to v_2; and since v_2, y ∈ G_2, there is a path from v_2 to y in G_2. These paths combined together must provide a path from x to y. This shows that G_1 ∪ G_2 is connected, a contradiction to the fact that G_1 and G_2 are two different connected components of a graph. Hence e_2 /∈ E(G_2) and e_1 ∈ E(G_1).

1.3 Lemma: If G = G_1 ⊕ G_2 where G_1 and G_2 are two connected graphs with V(G_1) ⊆ V(G_2) ⊆ V(G), then L(G) = L(G_1) ⊕ L(G_2).

Proof: Since G_1 ⊆ G, E(G_1) ⊆ E(G).

Also L(G) = L(G_1) ⊕ L(G_2). Hence E(L(G)) = E(G_1) ∪ E(G_2) = E(L(G_1)) ∪ E(L(G_2)) = E(L(G_1)) ⊆ E(L(G_2)).

In a similar way we get E(L(G_2)) ⊆ E(L(G_1)).

So L(G_1) ⊆ L(G) ⊆ L(G_2) ... (ii)

Let s ∈ E(L(G)). Then s = e_1e_2 for some e_1, e_2 ∈ V(L(G)) = E(G), and e_1, e_2 are adjacent in G.

By Lemma 1.2, e_1, e_2 ∈ E(G_1) or e_1, e_2 ∈ E(G_2), but not both. If e_1, e_2 ∈ E(G_1), then since e_1, e_2 are adjacent in G, e_1, e_2 are adjacent in G_1 and so s = e_1e_2 ∈ E(L(G_1)).

Similarly, if e_1, e_2 ∈ E(G_2), then s = e_1e_2 ∈ E(L(G_2)).

Hence E(L(G)) ⊆ E(L(G_1)) ∪ E(L(G_2)) ... (iii)

From (ii) and (iii), E(L(G)) = E(L(G_1)) ∪ E(L(G_2)) ... (iv)

Since G = G_1 ⊕ G_2, it follows that E(L(G_1) ∩ E(L(G_2)) = ∅.

Now from (i) and (iv), L(G) = L(G_1) ⊕ L(G_2), the proof is complete.

1.4 Example: Consider the graphs G_1 and G_2 given in Fig. 1 and Fig. 2 respectively.
Now let us construct the ring sum of \( L(G_1) \) and \( L(G_2) \).

\[ V(L(G_1) \oplus L(G_2)) = \{ e_1, e_2, f_1, f_2, f_3, f_4 \}, \]

\[ E(L(G_1) \oplus L(G_2)) = \{ \overline{e_1e_2}, \overline{f_1f_2}, \overline{f_1f_3}, \overline{f_1f_4}, \overline{f_2f_3}, \overline{f_2f_4} \}. \]

The graph \( L(G_1) \oplus L(G_2) \) is same as the graph given in Fig.6.

It is an easy observation that \( L(G) = L(G_1) \oplus L(G_3) \).

1.5 **Theorem:** If \( G \) is a graph consists of exactly \( m \) connected components \( G_1, G_2, \ldots, G_m \), then \( L(G) = (L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_m)) \).

**Proof:** The proof is by induction on \( m \).

If \( m = 2 \), then it follows through the above Lemma 1.3. Suppose that the result is true for \( m = k \).

Now take a graph \( G \) with \( m = k + 1 \) connected components \( G_1, G_2, \ldots, G_{k+1} \).

Now \( G = G_1 \oplus G_2 \oplus \ldots \oplus G_k \oplus G_{k+1} = (G_1 \oplus G_2 \oplus \ldots \oplus G_k) \oplus G_{k+1} \).

Now \( L(G) = L((G_1 \oplus G_2 \oplus \ldots \oplus G_k) \oplus G_{k+1}) \)

\[ = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k) \oplus L(G_{k+1}) \] (by Lemma 1.3)

\[ = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k) \oplus L(G_{k+1}) \] (by induction hypothesis), the proof is complete.

1.6 **Lemma:** If \( G \) is a connected graph, then \( L(G) \) is also a connected graph.

**Proof:** Let \( G \) be a connected graph. To show that \( L(G) \) is connected, let \( e_1, e_2 \in V(L(G)) = E(G) \).

Suppose \( e_1 = uv \) and \( e_2 = xy \) for some \( u, v, x, y \in V(G) \).

Since \( G \) is connected, there exists a path from \( v \) to \( x \). Suppose this path is \( v f_1 v_2 \ldots v_k x \) with \( v_k = x \).

Since \( f_1 \) is adjacent to \( f_2 \), \( f_2 \) is adjacent to \( f_3 \), \ldots \( f_{k-1} \) is adjacent to \( f_k \), it follows that \( \overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k} \) is a path from \( f_1 \) to \( f_k \) in \( L(G) \).

If \( e_1 = f_1 \) and \( e_2 = f_2 \), then \( \overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k} \) is a path from \( e_1 \) to \( e_2 \) in \( L(G) \).

If \( e_1 \neq f_1 \) and \( e_2 = f_2 \), then \( \overline{e_1f_1}, \overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k}, \overline{f_kf_2} \) is a path from \( e_1 \) to \( e_2 \) in \( L(G) \).

If \( e_1 \neq f_1 \) and \( e_2 \neq f_2 \), then \( \overline{e_1f_1}, \overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k}, \overline{f_kf_2}, \overline{e_2f_2} \) is a path from \( e_1 \) to \( e_2 \) in \( L(G) \).

Hence for any \( e_1, e_2 \in V(L(G)) \), there is a path between \( e_1 \) & \( e_2 \) in \( L(G) \). This shows that \( L(G) \) is connected, the proof is complete.

1.7 **Theorem:** The number of connected components in \( G \) is equal to the number of connected components in \( L(G) \).

**Proof:** Suppose the connected components of \( G \) are \( G_1, G_2, \ldots, G_k \).

Then \( G = G_1 \oplus G_2 \oplus \ldots \oplus G_k \).

By Theorem 1.6, \( L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k) \).
Since \( G \) is connected by Lemma1.6, the graph \( L(G) \) is connected and so \( L(G) \) is a connected component of \( L(G) \).

Hence \( L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k) \) and each \( L(G_i) \) is connected.

Thus the number of components of \( G = k = \) the number of components of \( L(G) \), the proof is complete.

1.8 Theorem: If \( G \) is a cycle of length \( n \), then \( L(G) \) is also a cycle of length \( n \).

Proof: Suppose \( G \) is a cycle of length \( n \).

Then \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( E(G) = \{e_1, e_2, \ldots, e_n\} \) with the cycle \( v_1e_1v_2e_2\ldots v_ne_1v_1 \).

Now \( V(L(G)) = E(G) = \{e_1, e_2, \ldots, e_n\} \).

Since \( e_1 \) and \( e_i \) for \( 2 \leq i \leq n \) are adjacent in \( G \), we get that \( e_iL(G) = \{e_1, e_2, \ldots, e_n\} \).

Thus \( e_1 e_2 e_3 \ldots e_n e_1 \in V(L(G)) \). The vertex set of \( Q_2(G) \) is defined as follows:

Thus \( L(G) \) is a cycle of length \( n \).

2. 1-QUASITOTAL GRAPHS

We start this section by introducing a new concept “1-quasitotal graph”.

2.1 Definition: Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). The 1-quasitotal graph, (denoted by \( Q_1(G) \)) of \( G \) is defined as follows:

The vertex set of \( Q_1(G) \), that is \( V(Q_1(G)) = V(G) \cup E(G) \).

Two vertices \( x, y \) in \( V(Q_1(G)) \) are adjacent if they satisfy one of the following conditions:

(i) \( x, y \) are in \( V(G) \) and \( xy \in E(G) \).

(ii) \( x, y \) are in \( E(G) \) and \( x, y \) are incident in \( G \).

2.2 Note: (i) \( G \) is a subgraph of \( Q_1(G) \); and

(ii) \( Q_1(G) \) is a subgraph of \( T(G) \).

2.3 Example: Consider the graph \( G \) given in Fig. 7. Let us construct the 1-quasitotal graph \( Q_1(G) \) of the graph \( G \).

V(Q_1(G)) = \{V(G) \cup E(G)\} = \{v_1, v_2, v_3, e_1, e_2, e_3, e_4\}

It is clear that \( E(G) \subseteq E(Q_1(G)) \).

So \( v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in E(Q_1(G)) \).

Since \( e_1 \) and \( e_2 \) are incident in \( G \), there is an edge \( e_1e_2 \in E(Q(G)) \).

Therefore \( E(Q_1(G)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, e_1e_2, e_2e_3, e_3e_4\} \).

The 1-quasitotal graph \( Q_1(G) \) is given by the Fig. 8.

2.4 Theorem: \( Q_1(G) = G \oplus L(G) \).

Proof: By the definition of \( Q_1(G) \),

\[ V(Q_1(G)) = V(G) \cup E(G) = V(G) \cup V(L(G)) \] (since \( V(L(G)) = E(G) \)).

Let \( s \in E(G) \). If \( s \) is an edge in \( G \), then \( s \in E(G) \).

If \( s \notin E(G) \), then (by the definition of \( Q_1(G) \)) \( s = e_1e_2 \) where \( e_1, e_2 \in E(G) \) and \( e_1, e_2 \) are adjacent edges in \( G \). By the definition of \( L(G) \) it follows that \( s \in V(L(G)) \).

Therefore \( E(Q_1(G)) \subseteq E(G) \cup E(L(G)) \). By Note 2.2, \( E(G) \cup E(L(G)) \subseteq E(Q_1(G)) \).

Hence \( Q_1(G) = G \cup L(G) \), the union of the two graphs \( G \) and \( L(G) \).

Since \( V(G) \cap V(L(G)) = V(G) \cap E(G) = \emptyset \), there exists no common edge in \( G \) and \( L(G) \). This means that \( E(G) \cap E(L(G)) = \emptyset \).

This implies that \( G \cup L(G) = G \oplus L(G) \).

Hence \( Q_1(G) = G \cup L(G) = G \oplus L(G) \), the proof is complete.

2.5 Corollary: If \( G \) is a cycle of length \( n \), then \( Q_1(G) \) is the ring sum of exactly two disjoint cycles of length \( n \).

Proof: Suppose \( G \) is a cycle of length \( n \).

By Theorem1.8, \( L(G) \) is a cycle of length \( n \).

Since \( Q_1(G) = G \oplus L(G) \) (by Theorem 2.4) \( Q_1(G) \) is equal to the ring sum of two disjoint cycles of length \( n \), the proof is complete.

3. 2-QUASITOTAL GRAPHS

We start this section by introducing a new concept “2-quasitotal graph”.

3.1 Definition: Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \).

The 2-quasitotal graph of \( G \), denoted by \( Q_2(G) \) is defined as follows: The vertex set of \( Q_2(G) \), that is, \( V(Q_2(G)) = V(G) \cup E(G) \).
E(G). Two vertices x, y in V(Q_2(G)) are adjacent in Q_2(G) in case one of the following holds:

(i) x, y are in V(G) and \( \overline{xy} \in E(G) \).

(ii) x is in V(G); y is in E(G); and x, y are incident in G.

3.2 Note: (i) G is a subgraph of Q_2(G); and

(ii) Q_2(G) is a subgraph of T(G).

3.3 Example: Consider the graph given in fig 9.

![Fig 9](image)

Let us construct Q_2(G) of G.

\[ V(Q_2(G)) = V(G) \cup E(G), \]
\[ E(Q_2(G)) = \{ v_1v_2, v_2v_3, v_2v_4, v_2e_1, v_1v_2, e_1v_2, e_1v_3, e_2v_3, e_3v_4, e_2e_3, e_2v_4, e_3v_4 \}. \]

![Fig 10](image)

The Q_2(G) is given in Fig 10

3.4 Lemma: If e = \( \overline{uv} \in E(G) \), then there exist a triangle in E(Q_2(G)) containing e as one of the edges.

Proof: Suppose G is a graph with |E(G)| = 1. Let e \in E(G) and \( e = \overline{vu} \) for some v, u \in V(G). Now e, v, u \in V(G) \cup E(G) = V(Q_2(G)).

Now \( \overline{vu} \in E(G) \subseteq E(Q_2(G)). \) Since e and u are incident in G, we have that \( \overline{uv} \in E(Q_2(G)). \) Since e and v are incident in G, we have that \( \overline{ev} \in E(Q_2(G)). \)

So \( \overline{vu}, \overline{ue}, \overline{ev} \in E(Q_2(G)) \) and these edges put together form a triangle, the proof is complete.

3.5 Lemma: If e is not in a triangle of G and e = \( \overline{vu} \in E(G) \) is only the edge between the vertices u and v in G, then there is only one triangle in E(Q_2(G)) containing e as one of the edges.

Proof: By Lemma 3.4, we know that \( \overline{uv}, \overline{ve}, \overline{eu} \) is a triangle (in Q_2(G)) containing the edge e = \( \overline{uv} \).

Let \( \overline{ab}, \overline{bc}, \overline{ca} \) be a triangle in Q_2(G) containing e = \( \overline{uv} \). Without loss of generality, we assume that \( \overline{ab} = e \) and so \( \overline{ab} = \overline{uv} \). Further, we assume that a = u and b = v.

If \( \overline{bc} \) and \( \overline{ca} \) are edges in G, then \( \overline{ab}, \overline{bc}, \overline{ca} \) is a triangle in G containing e, a contradiction to our hypothesis. So \( \overline{bc} \) or \( \overline{ca} \) is not in E(G).

Suppose \( \overline{bc} \) is not in E(G). Since b \in V(G) it follows that c \in E(G).

Since \( \overline{bc} \) is in E(Q_2(G)), by the definitions of Q_2(G), it follows that the edge c of G is incident on the vertex b = v of G.

Now \( \overline{ca} \in E(Q_2(G)) \) implies that the edge c is incident on the vertex a = u.

Thus c is an edge between its end points u and v.

Since there is only one edge (in G) between the vertices u and v, we get c = e.

Thus the triangle \( \overline{ab}, \overline{bc}, \overline{ca} \) taken in E(Q_2(G)) is nothing but \( \overline{uv}, \overline{ve}, \overline{eu} \). Hence there is only one triangle in Q_2(G) containing (or corresponding to) e.

3.6 Note: If \( \overline{xv} \in E(Q_2(G)) \setminus E(G) \), then by the definition of Q_2(G) it follows that one of the x, y is edge (say x) in G and the edge x is incident on the vertex y (of G) in G.

3.7 Lemma: Every triangle in Q_2(G) contains an edge of G.

Proof: Let \( \overline{ab}, \overline{bc}, \overline{ca} \) be a triangle in Q_2(G).

If possible suppose that neither \( \overline{ab} \) nor \( \overline{bc} \) nor \( \overline{ca} \) is an edge in G.

Since \( \overline{ab} \) is an edge in E(Q_2(G)) \setminus E(G), one of the a, b is an edge and the other is a vertex in G.

Without loss of generality, assume that \( a \in E(G) \) and b \in V(G) and a is incident on b.

Since b \in V(G), \( \overline{bc} \in E(Q_2(G)) \setminus E(G) \), we have that \( c \in E(G) \) and c is incident on b.

Since \( \overline{ca} \in E(Q_2(G)) \setminus E(G) \) and \( c \in E(G) \), it follows that \( a \in V(G) \) and c is incident on a. This fact \( a \in V(G) \) is a contradiction to the fact \( a \in E(G) \).

Thus one of the \( \overline{ab}, \overline{bc}, \overline{ca} \) is an edge in G.

3.8 Theorem: If G is a graph containing only one edge (that is |E(G)| = 1) then the graph Q_2(G) contains unique triangle.

Proof: (Existence): Let \( E(G) = \{ e \} \) and u, v \in V(G) with e = \( \overline{uv} \). By Lemma 3.4,

\( \overline{uv}, \overline{ve}, \overline{eu} \) is a triangle in Q_2(G) containing the edge e.

(Uniqueness): Since e is only the edge in G, the graph G contains no triangles. So the statement "e" is not in any triangle of "G" is true.

Thus by using Lemma 3.5, we can conclude that Q_2(G) contains only one triangle containing "e"... (i)
Now we verify that any triangle in Q2(G) contains e.

Let \( xy \), \( xz \), \( zx \) be a triangle in Q2(G).

By Lemma 3.7, this triangle contains an edge of G. Since G contains only one edge e, it follows that the triangle \( (xy) \), \( (xz) \), \( (zx) \) contains e. From the above steps, every triangle in Q2(G) contains the edge “e” \( \ldots \) (ii)

From (i) & (ii), we get that Q2(G) contains unique triangle.

3.9 Lemma: Suppose G contains two distinct edges \( e_1 = uv \) and \( e_2 = xy \) \( \square \) If \( \{u, v\} \cap \{x, y\} = \emptyset \), then Q2(G) contains two distinct triangles one containing \( e_1 \) and other containing \( e_2 \). Moreover there is no common vertex between two triangles.

(ii) If \( \{u, v\} \cap \{x, y\} \neq \emptyset \), then Q2(G) contains two distinct triangles one containing \( e_1 \) and other containing \( e_2 \). Moreover if \( \{u, v\} \cap \{x, y\} \neq \emptyset \) then a is a common vertex to these two triangles.

Proof: Given that \( e_1 = uv \) and \( e_2 = xy \) are two edges in G.

By Lemma 3.4, \( u\overline{e_1}, \overline{e_2}, \overline{vu} \) is a triangle in Q2(G) containing \( e_1 = uv \); and \( x\overline{e_2}, \overline{e_2y}, \overline{xy} \) is a triangle in Q2(G) containing \( e_2 = xy \).

Clearly these are two distinct triangles.

Suppose that \( \{u, v\} \cap \{x, y\} = \emptyset \).

If possible suppose that the triangles \( \{u\overline{e_1}, \overline{e_1y}, \overline{vu} \} \) and \( \{x\overline{e_2}, \overline{e_2y}, \overline{xy} \} \) have a common vertex.

The vertex sets of these triangles are \{u, v, e1\} and \{x, y, e2\}.

Since \( e_1 \neq e_2 \) (as edges in G), \( e_1 \neq e_2 \) (as vertices in Q2(G)).

The remaining part is clear because \( \{u, v\} \cap \{x, y\} = \emptyset \).

Hence there is no common vertex between the two triangles.

(ii) Suppose \( \{u, v\} \cap \{x, y\} \neq \emptyset \). If \( \{u, v\} = \{x, y\} \), then \( e_1 = xy = e_2 \), a contradiction. So \( \{u, v\} \neq \{x, y\} \) and \( \{u, v\} \cap \{x, y\} \neq \emptyset \).

Without loss of generality, assume that u = x and v \( \neq \emptyset \).

In this case, the vertex sets of the triangles are \{u, v, e1\} and \{x, y, e2\}.

This shows that the two triangles have a common vertex u, the proof is complete.

3.10 Theorem: Let G be a graph. Then the following conditions are equivalent:

(i) \( |E(G)| = 1 \);

(ii) Q2(G) contains unique triangle.

Proof: (i) \( \Rightarrow \) (ii): Theorem 3.8

(ii) \( \Rightarrow \) (i): Suppose Q2(G) contains unique triangle. By Lemma 3.7, every triangle of Q2(G) contains at least one edge of G. Since Q2(G) contains a triangle, \( |E(G)| \geq 1 \). If \( |E(G)| > 1 \), then E(G) contains two distinct edges. By Lemma 3.9, it follows that Q2(G) contains two distinct triangles, a contradiction to (ii).

Thus \( |E(G)| = 1 \), the proof is complete.

4. CONCLUSIONS

There is a scope for concepts of total graphs, quasi-total graphs can be extended to finite directed graph with suitable assumptions.

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6. REFERENCES


