Semi-Star-Alpha-Open Sets and Associated Functions

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ABSTRACT: The aim of this paper is to introduce various functions associated with semi*α-open sets. Here semi*α-continuous, semi*α-irresolute, contra-semi*α-continuous and contra-semi*α-irresolute functions are defined. Characterizations for these functions are given. Further their fundamental properties are investigated. Many other functions associated with semi*α-open sets and their contra versions are introduced and their properties are studied. In addition strongly semi*α-irresolute functions, contra-strongly semi*α-irresolute functions, semi*α-totally continuous, totally semi*α-continuous functions and semi*α-homeomorphisms are introduced and their properties are investigated.

General Terms: General topology

Keywords: semi*α-continuous, semi*α-irresolute, semi*α-open, semi*α-closed, pre-semi*α-open function, pre-semi*α-closed function

1. INTRODUCTION

In this paper various functions associated with semi*α-open sets are introduced and their properties are investigated.

2. PRELIMINARIES
Throughout this paper X, Y and Z will always denote topological spaces on which no separation axioms are assumed.

Definition 2.1[10]: A subset A of a topological space (X, τ) is called (i) generalized closed (briefly g-closed) if Cl(A) ⊆ U whenever A ⊆ U and U is open .

Definition 2.2: Let A be a subset of X. Then (i) generalized closure[11] of A is defined as the intersection of all g-closed sets containing A and is denoted by Cl*_{g}(A).

(ii) generalized interior of A is defined as the union of all g-open subsets of A and is denoted by Int_{g}(A).

Definition 2.3: A subset A of a topological space (X, τ) is (i) semi-open [1] (resp. α-open[12], semi α-open[13], semi-preopen[14], semi-open, semi*α-open[7], semi*-preopen[6]) if A ⊆ Cl(Int(A)) (resp. A ⊆ Int(Cl(Int(A))), A ⊆ Int(Cl(Cl(Int(A))), A ⊆ Cl(Int(Cl(A))), A ⊆ Cl*_{g}(Int(A)), A ⊆ Cl*_{g}(Int(Int(A))).

Definition 2.4: Let A be a subset of X. Then (i) The semi*α-interior [7] of A is defined as the union of all semi*α-open subsets of A and is denoted by s*αInt(A).

Theorem 2.8: [7] (i) Every α-open set is semi*α-open .

Theorem 2.10: [7] (i) Arbitrary union of semi*α-open sets is also semi*α-open.

Remark 2.9:[8] Similar results for semi*α-closed sets are also true.
(ii) If $A$ is semi*-α-open in $X$ and $B$ is open in $X$, then $A \cap B$ is semi*-α-open in $X$.

(iii) A subset $A$ of a space $X$ is semi*-α-open if and only if $s^*\alpha int(A) = A$.

**Theorem 2.11:** [7] For a subset $A$ of a space $X$ the following are equivalent:

(i) $A$ is semi*-α-open in $X$

(ii) $A \subseteq C^*(s^*\alpha int(A))$

(iii) $C^*(s^*\alpha int(A)) \subseteq C^*(A)$.

**Theorem 2.12:** [8] For a subset $A$ of a space $X$ the following are equivalent:

(i) $A$ is semi*-α-closed in $X$.

(ii) $Int^*(\alpha Cl(A)) = A$

(iii) $Int^*(\alpha Cl(A)) = Int^*(A)$.

**Theorem 2.13:** [8] (i) A subset $A$ of a space $X$ is semi*-α-closed if and only if $s^*\alpha Cl(A) = A$.

(ii) Let $\Delta X$ and let $x \in X$. Then $x \in s^*\alpha Cl(A)$ if and only if every semi*-α-open set in $X$ containing $x$ intersects $A$.

**Definition 2.14:** [9] If $A$ is a subset of $X$, the semi*-α-Frontier of $A$ is defined by $s^*\alpha Fr(A) = s^*\alpha Cl(A) \setminus s^*\alpha int(A)$.

**Theorem 2.15:** [9] If $A$ is a subset of $X$, then $s^*\alpha Fr(A) = s^*\alpha Cl(A) \setminus s^*\alpha int(A)$.

**Definition 2.16:** [15] A function $f: X \to Y$ is said to be (i) semi α*-continuous (resp. semi α**-continuous ) if $f^{-1}(V)$ is semi α-open (resp. open) set in $X$ for every semi α-open set $V$ in $Y$.

(ii) totally semi-continuous [16] if $f^{-1}(V)$ is semi regular in $X$ for every open set $V$ in $Y$.

(iii) semi-totally continuous [17] if $f^{-1}(V)$ is clopen in $X$ for every semi-open set $V$ in $Y$.

3. **SEMI*-α-CONTINUOUS FUNCTIONS**

In this section we define the semi*-α-continuous and contra-semi*-α-continuous functions and investigate their fundamental properties.

**Definition 3.1:** A function $f: X \to Y$ is said to be semi*-α-continuous at $x \in X$ if for each open set $V$ in $Y$ containing $f(x)$, there is a semi*-α-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$.

**Definition 3.2:** A function $f: X \to Y$ is said to be semi*-α-continuous if $f^{-1}(V)$ is semi*-α-open in $X$ for every open set $V$ in $Y$.

**Theorem 3.3:** Let $f: X \to Y$ be a function. Then the following statements are equivalent:

(i) $f$ is semi*-α-continuous.

(ii) $f$ is semi*-α-continuous at each point $x \in X$.

(iii) $f^{-1}(F)$ is semi*-α-closed in $X$ for every closed set $F$ in $Y$.

(iv) $s^*\alpha Cl(f(A)) \subseteq Cl(s^*\alpha f(A))$ for every subset $A$ of $X$.

(v) $s^*\alpha Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every subset $B$ of $Y$.

(vi) $Int(f^{-1}(B)) \subseteq s^*\alpha Int(f^{-1}(B))$ for every subset $B$ of $Y$.

(vii) $Cl(s^*\alpha (f^{-1}(F))) = Int(s^*\alpha f^{-1}(F))$ for every closed set $F$ in $Y$.

(viii) $C^*(s^*\alpha Int(f^{-1}(V))) = Int(f^{-1}(V))$ for every open set $V$ in $Y$.

**Proof:** (i) $\Rightarrow$ (iii): Let $f: X \to Y$ be semi*-α-continuous. Let $x \in X$ and $V$ be an open set in $Y$ containing $f(x)$. Then $x \notin f^{-1}(V)$. Since $f$ is semi*-α-continuous, $U = f^{-1}(V)$ is a semi*-α-open set in $X$ containing $x$ such that $f(U) \subseteq V$.

(ii) $\Rightarrow$ (i): Let $f: X \to Y$ be semi*-α-continuous at each point of $X$. Let $x \notin f^{-1}(V)$. Then $V$ is an open set in $Y$ containing $f(x)$. By (ii), there is a semi*-α-open set $U$, in $X$ containing $x$ such that $x \notin U \subseteq f^{-1}(V)$. Therefore $U \notin f^{-1}(V)$. Hence $f^{-1}(V) = U \cup f^{-1}(V)$. By Theorem 2.10(i), $f^{-1}(V)$ is semi*-α-open in $X$.

(iii) $\Rightarrow$ (ii): Let $C$ be a closed set in $Y$. Then $C \cap f^{-1}(V)$ is open in $Y$. Then $f^{-1}(V)$ is semi*-α-open in $X$. Therefore $f^{-1}(V) \notin f^{-1}(C)$. Hence $f^{-1}(V) \notin C$. By Theorem 2.10(i), $f^{-1}(V)$ is semi*-α-open in $X$.

(iv) $\Rightarrow$ (iv): Let $A \subseteq X$. Then $A \subseteq Y$. Therefore $f \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is semi*-α-open in $X$.

(v) $\Rightarrow$ (v): Let $V$ be an open set in $X$. Then $f^{-1}(V)$ is semi*-α-open. By (vii), $f^{-1}(V)$ is semi*-α-closed. Hence $f^{-1}(V) = f^{-1}(f^{-1}(V))$ is semi*-α-open.

(vi) $\Rightarrow$ (vi): Let $A \subseteq X$. Then $A \subseteq Y$. Therefore $f \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is semi*-α-open in $X$.

(vi) $\Rightarrow$ (vii): Let $B \subseteq Y$. Then $A \subseteq B$. By assumption, $f(s^*\alpha Cl(A)) \subseteq s^*\alpha Cl(B)$. This implies that $s^*\alpha Cl(A) \subseteq f^{-1}(Cl(B))$.

(vii) $\Rightarrow$ (viii): Let $A \subseteq X$. Then $A \subseteq B$. Therefore $f \subseteq f^{-1}(B)$. Hence $s^*\alpha Cl(f^{-1}(B)) = f^{-1}(B)$.

(viii) $\Rightarrow$ (vi): Every semi*-α-continuous function is semi*-α-continuous.

(vii) $\Rightarrow$ (vi): Every semi*-α-continuous function is semi*-α-continuous.

(vii) $\Rightarrow$ (vii): Every semi*-α-continuous function is semi*-α-continuous.

Remark 3.5: In general the converse of each of the statements in Theorem 3.4 is not true.

**Theorem 3.6:** If the topology of the space $Y$ is given by a basis $B$, then a function $f: X \to Y$ is semi*-α-continuous if and only if the inverse image of every basic open set in $Y$ under $f$ is semi*-α-open.

**Proof:** Suppose $f: X \to Y$ is semi*-α-continuous. Then inverse image of every open set in $Y$ is semi*-α-open in $X$. In particular, inverse image of every basic open set in $Y$ is semi*-α-open in $X$. Conversely, let $V$ be an open set in $Y$. Then $f^{-1}(V)$ is semi*-α-open in $X$. Hence $f^{-1}(B)$ is semi*-α-open for each $B$. By Theorem 2.10(i), $f^{-1}(V)$ is semi*-α-open in $X$.

**Theorem 3.7:** A function $f: X \to Y$ is not semi*-α-continuous at point $x \in X$ if and only if $x$ belongs to the semi*-α-frontier of the inverse image of some open set in $Y$ containing $f(x)$.

**Proof:** Suppose $f$ is not semi*-α-continuous at $x$. Then by Definition 3.1, there is an open set $V$ in $Y$ containing $f(x)$ such that $f(U)$ is not a subset of $V$ for every semi*-α-open set $U$ in $X$ containing $x$. Hence $U \cap f^{-1}(V) \neq \emptyset$ for every semi*-α-open set $U$ containing $x$. By Theorem 2.13(ii), we get $x \in s^*\alpha Cl(f^{-1}(V))$. Also $x \in f^{-1}(V) \subseteq s^*\alpha Cl(f^{-1}(V))$. Hence $x \in s^*\alpha Cl(f^{-1}(V))$. 

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Theorem 3.8: Let \( f : X \to \Pi X \) be semi-\( \alpha \)-continuous where \( \Pi X \) is given the product topology and \( f(x)=\pi (\pi (x)) \). Then each coordinate function \( \pi _j X \to X \) is semi-\( \alpha \)-continuous.

Proof: Let \( V \) be an open set in \( \Pi X \), Then \( f \circ \pi _j X \to X \) is semi-\( \alpha \)-continuous.

Theorem 3.9: Let \( f : X \to Y \) be defined by \( f(x)=f(x) \) and \( \Pi X \) be given the product topology. Suppose \( S \ast \alpha (O) \) is closed under finite intersection. Then \( f \) is semi-\( \alpha \)-continuous if each coordinate function \( f \circ \pi _j X \to X \) is semi-\( \alpha \)-continuous.

Proof: Let \( V \) be a basic open set in \( \Pi X \). Then \( V=\bigcap \pi _j \circ x \in V \) where each \( V \) is open in \( X \), the intersection being taken over finitely many \( \alpha \). Now \( f^{-1}(V)=f^{-1}(\pi _j \circ x \in V)=\bigcap \pi _j \circ x \in V \) is semi-\( \alpha \)-open, by hypothesis. Hence by Theorem 3.6, \( f \) is semi-\( \alpha \)-continuous.

Theorem 3.10: Let \( f : X \to Y \) be continuous and \( g : Z \to X \) be semi-\( \alpha \)-open. Let \( h = X \to Y \) be defined by \( h(x)=f(x) \), \( g(x) \) and \( Y \times Z \) be given the product topology. Then \( h \) is semi-\( \alpha \)-continuous.

Proof: By virtue of Theorem 3.6, it is sufficient to show that inverse image under \( h \) of every basic open set in \( Y \times Z \) is semi-\( \alpha \)-open in \( X \). Let \( U \times V \) be a basic open set in \( Y \times Z \). Then \( h^{-1}(U \times V)=f^{-1}(V) \times g^{-1}(U) \) is continuous, \( f^{-1}(V) \) open in \( Y \), and \( g^{-1}(U) \) is semi-\( \alpha \)-open in \( X \). By invoking Theorem 2.10(ii), we get \( h^{-1}(U \times V)=f^{-1}(V) \times g^{-1}(U) \) is semi-\( \alpha \)-open.

Remark 3.11: The above theorem is true even if \( f \) is semi-\( \alpha \)-continuous and \( g \) is continuous.

Theorem 3.12: Let \( f : X \to Y \) be semi-\( \alpha \)-continuous and \( g : Y \to Z \) be continuous. Then \( g \circ f : X \to Z \) is semi-\( \alpha \)-continuous.

Proof: Let \( V \) be an open set in \( Z \). Since \( g \) is continuous, \( g^{-1}(V) \) is open in \( X \). By semi-\( \alpha \)-continuity of \( f \), \( f^{-1}(g^{-1}(V)) \) is semi-\( \alpha \)-open in \( X \). Hence \( g \circ f \) is semi-\( \alpha \)-continuous.

Remark 3.13: Composition of two semi-\( \alpha \)-continuous functions need not be semi-\( \alpha \)-continuous.

Definition 3.14: A function \( f : X \to Y \) is called contra-semi-\( \alpha \)-continuous if \( f^{-1}(V) \) is semi-\( \alpha \)-closed in \( X \) for every open set \( V \) in \( Y \).

Theorem 3.15: For a function \( f : X \to Y \), the following are equivalent:

(i) \( f \) is contra-semi-\( \alpha \)-continuous.

(ii) For each \( x \in X \) and each closed set \( F \) in \( Y \), there exists a semi-\( \alpha \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq F \).

(iii) The inverse image of each closed set in \( Y \) is semi-\( \alpha \)-open in \( X \).

(iv) \( C^\alpha (\text{Cl}_{\alpha}(f^{-1}(V)))=C^\alpha (f^{-1}(V)) \) for every closed set \( F \) in \( Y \).

(v) \( \text{Int}^\alpha (\text{Cl}_{\alpha}(f^{-1}(V)))=\text{Int}^\alpha (f^{-1}(V)) \) for every open set \( V \) in \( Y \).

Proof: (i) \( \Rightarrow \) (ii): Let \( f : X \to Y \) be contra-semi-\( \alpha \)-continuous. Then \( x \in X \) and \( F \) is a closed set in \( Y \) containing \( f(x) \). Then \( V \subseteq f^{-1}(F) \) is an open set in \( Y \) not containing \( f(x) \). Since \( f \) is contra-semi-\( \alpha \)-continuous, \( f^{-1}(V) \) is a semi-\( \alpha \)-closed set in \( X \) not containing \( x \). That is, \( f^{-1}(F) \) is a semi-\( \alpha \)-closed set in \( X \) not containing \( x \). Therefore \( f^{-1}(V) \) is a semi-\( \alpha \)-open set in \( X \) containing \( x \) such that \( f(U) \subseteq F \).

(ii) \( \Rightarrow \) (iii): Let \( F \) be a closed set in \( Y \). Let \( x \in \text{Int}^\alpha (f^{-1}(F)) \), then \( f(x) \in F \). By (ii), there is a semi-\( \alpha \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq F \). That is, \( x \in U \subseteq f^{-1}(F) \). Therefore \( f^{-1}(F) \) is a semi-\( \alpha \)-open set in \( X \) containing \( x \).

(iii) \( \Rightarrow \) (iv): Let \( f \) be a closed set in \( Y \). By (iii), \( f^{-1}(F) \) is a semi-\( \alpha \)-open set in \( X \). By Theorem 2.11(i), \( f^{-1}(F) \) is semi-\( \alpha \)-open in \( X \).

(iv) \( \Rightarrow \) (v): If \( V \) is any open set in \( Y \), then \( Y \times V \) is closed in \( Y \). By (iv), we have \( C^\alpha (\text{Cl}_{\alpha}(f^{-1}(V)))=C^\alpha (f^{-1}(V)) \). Taking the complements, we get \( \text{Int}^\alpha (\text{Cl}_{\alpha}(f^{-1}(V)))=\text{Int}^\alpha (f^{-1}(V)) \).

(v) \( \Rightarrow \) (i): Let \( V \) be any open set in \( Y \). Then by assumption, \( \text{Int}^\alpha (\text{Cl}_{\alpha}(f^{-1}(V)))=\text{Int}^\alpha (f^{-1}(V)) \). By Theorem 2.12, \( f^{-1}(V) \) is semi-\( \alpha \)-closed.

Theorem 3.16: Every contra-semi-\( \alpha \)-continuous function is contra-semi-\( \alpha \)-continuous.

Proof: Follows from Remark 2.9.

Remark 3.17: It can be seen that the converse of the above theorem is not true.

Theorem 3.18: Every contra-semi-\( \alpha \)-continuous function is contra-semi-\( \alpha \)-continuous.

Proof: Let \( f : X \to Y \) be contra-semi-\( \alpha \)-continuous. Let \( V \) be a closed set in \( Y \). Since \( f \) is contra-semi-\( \alpha \)-continuous, \( f^{-1}(V) \) is semi-\( \alpha \)-closed in \( X \). By Remark 2.9, \( f^{-1}(V) \) is semi-\( \alpha \)-closed in \( X \). Hence \( f \) is contra-semi-\( \alpha \)-continuous.

Remark 3.19: It can be easily seen that the converse of the above theorem is not true.

Composition of two contra-semi-\( \alpha \)-continuous functions need not be contra-semi-\( \alpha \)-continuous.

4. SEMI-\( \alpha \)-IRRESOLUTE FUNCTIONS

In this section we define the semi-\( \alpha \)-irresolute and contra-semi-\( \alpha \)-irresolute functions and investigate their fundamental properties.

Definition 4.1: A function \( f : X \to Y \) is said to be semi-\( \alpha \)-irresolute at \( x \in X \) if for each semi-\( \alpha \)-open set \( V \) in \( Y \), \( f \) is a semi-\( \alpha \)-open set \( U \) in \( X \) such that \( f(U) \subseteq Y \).

Definition 4.2: A function \( f : X \to Y \) is said to be semi-\( \alpha \)-irresolute if \( f^{-1}(V) \) is semi-\( \alpha \)-open in \( X \) for every semi-\( \alpha \)-open set \( V \) in \( Y \).

Definition 4.3: A function \( f : X \to Y \) is said to be contra-semi-\( \alpha \)-irresolute if \( f^{-1}(V) \) is semi-\( \alpha \)-closed in \( X \) for every semi-\( \alpha \)-closed set \( V \) in \( Y \).
Definition 4.4: A function $f : X \to Y$ is said to be strongly semi-$\alpha$-irresolute if $f^{-1}(V)$ is open in $X$ for every semi-$\alpha$-open set $V$ in $Y$.

Definition 4.5: A function $f : X \to Y$ is said to be contra-
strongly semi-$\alpha$-irresolute if $f^{-1}(V)$ is closed in $X$ for every semi-
$\alpha$-open set $V$ in $Y$.

Theorem 4.6: Every semi-$\alpha$-irresolute function is semi-$\alpha$-
continuous.
Proof: Let $f : X \to Y$ be semi-$\alpha$-irresolute. Let $V$ be open in $Y$. Then by Theorem 2.8(ii), $V$ is semi-$\alpha$-open. Since $f$ is semi-$\alpha$-irresolute, $f^{-1}(V)$ is semi-$\alpha$-open in $X$. Thus $f$ is semi-$\alpha$-continuous.

Theorem 4.7: Every constant function is semi-$\alpha$-irresolute.
Proof: Let $f : X \to Y$ be a constant function defined by $f(x) = y_0$ for all $x \in X$, where $y_0$ is a fixed point in $Y$. Let $V$ be a semi-$\alpha$-open set in $Y$. Then $f^{-1}(V) = \{x \in X: f(x) \in V\}$ or $y_0 \notin V$. Thus $f^{-1}(V)$ is semi-$\alpha$-open in $X$. Hence $f$ is semi-$\alpha$-irresolute.

Theorem 4.8: Let $f : X \to Y$ be a function. Then the following are equivalent:
(i) $f$ is semi-$\alpha$-irresolute.
(ii) $f$ is semi-$\alpha$-irresolute at each point of $X$.
(iii) $f^{-1}(F)$ is semi-$\alpha$-closed in $X$ for every semi-$\alpha$-closed set $F$ in $Y$.
(iv) $f(s*aCl(A)) \subseteq s*aCl(f(A))$ for every subset $A$ of $X$.
(v) $f(s*aCl(f^{-1}(B))) \subseteq s*aCl(f(B))$ for every subset $B$ of $Y$.
(vi) $\text{Int}(s*aCl(f^{-1}(F))) = \text{Int}(f^{-1}(F))$ for every semi-$\alpha$-closed set $F$ in $Y$.
(vii) $\text{Cl}(\text{Int}(f^{-1}(V))) = \text{Cl}(f^{-1}(V))$ for every semi-$\alpha$-open set $V$ in $Y$.
Proof: (i) $\Rightarrow$ (ii): Let $f : X \to Y$ be semi-$\alpha$-irresolute. Let $x \in X$ and $V$ be a semi-$\alpha$-open set in $Y$ containing $f(x)$. Then $x \in f^{-1}(V)$. Since $f$ is semi-$\alpha$-irresolute, $U = f^{-1}(V)$ is a semi-$\alpha$-
open set in $X$ containing $x$ such that $f(U) \subseteq V$.
(ii) $\Rightarrow$ (i): Let $f : X \to Y$ be semi-$\alpha$-irresolute at each point of $X$. Let $V$ be a semi-$\alpha$-open set in $Y$. Let $x \in f^{-1}(V)$. Then $V$ is a semi-$\alpha$-open set in $Y$ containing $f(x)$. By (ii), there is a semi-$\alpha$-open set $U$ in $X$ containing $x$ such that $x \in U \subseteq V$. Therefore $U \subseteq \text{Cl}(f^{-1}(V))$. Hence $f^{-1}(V) = U \cup \{x \in X : f^{-1}(V)\}$. By Theorem 2.10(i), $f^{-1}(V)$ is semi-$\alpha$-open in $X$.
(iii) $\Rightarrow$ (i): Let $F$ be a semi-$\alpha$-closed set in $Y$. Then $V = Y \setminus F$ is semi-$\alpha$-open in $Y$. Then $f^{-1}(V)$ is semi-$\alpha$-open in $X$. Therefore $f^{-1}(V) = f^{-1}(V) \setminus Y \setminus F = f^{-1}(V)$ is semi-$\alpha$-closed.
(iv) $\Rightarrow$ (iii): Let $A \subseteq X$. Let $f$ be a semi-$\alpha$-closed set containing $A$. This implies that $s*aCl(A) \subseteq f^{-1}(F)$ and hence $s*aCl(f(A)) \subseteq f(F)$. Therefore $f(s*aCl(f(A))) \subseteq f(A)$.
(v) $\Rightarrow$ (iv): Let $BGY$ and let $A \subseteq f(A)$. By assumption, $f(s*aCl(A)) \subseteq s*aCl(f(A)) \subseteq s*aCl(B)$. This implies that $s*aCl(f^{-1}(F)) \subseteq s*aCl(A)$. Hence $s*aCl(f^{-1}(F)) = f^{-1}(s*aCl(B))$.
(vi) $\Rightarrow$ (v): Let $f$ be semi-$\alpha$-closed in $Y$. Then $s*aCl(f) = F$. Therefore $f^{-1}(V) = f^{-1}(V) \subseteq f^{-1}(F)$, Hence $s*aCl(f^{-1}(F)) = f^{-1}(V)$. By Theorem 2.13(i), $f^{-1}(F)$ is semi-$\alpha$-closed.
(vii) $\Rightarrow$ (vi): The equivalence of (v) and (vi) can be proved by taking the complements.
(viii) $\Rightarrow$ (i): Follows from Theorem 4.2.
(ix) $\Rightarrow$ (ii): Follows from Theorem 2.11.

Theorem 4.9: Let $f : X \to Y$ be a function. Then $f$ is not semi-$\alpha$-irresolute at a point $x$ in $X$ if and only if $x$ belongs to the semi-$\alpha$-frontier of the inverse image of some semi-$\alpha$-open set in $Y$ containing $f(x)$.
Proof: Suppose $f$ is not semi-$\alpha$-irresolute at $x$. Then by Definition 4.1, there is a semi-$\alpha$-open set $V$ in $Y$ containing $f(x)$ such that $f(U) \subseteq V$ is a semi-$\alpha$-open set $U$ in $X$ containing $x$. Hence $f^{-1}(U) \cap V \neq \emptyset$ for every semi-$\alpha$-open set $U$ in $X$ containing $x$. Thus $x \in f^{-1}(V) \subseteq s*aCl(f^{-1}(V))$. Hence by Theorem 2.15, $x \notin s*aCl(f^{-1}(V))$. On the other hand, let $f$ be semi-$\alpha$-irresolute at $x$. Let $V$ be a semi-$\alpha$-open set in $Y$ containing $f(x)$. Then there is a semi-$\alpha$-open set $U$ in $X$ containing $x$ such that $f(x) \in f(U) \subseteq V$. Therefore $U \subseteq f^{-1}(V)$. Hence $x \notin s*aCl(f^{-1}(V))$. Therefore by Definition 2.14, $x \notin s*aCl(f^{-1}(V))$ for every open set $V$ containing $f(x)$.

Theorem 4.10: Every contra-semi-$\alpha$-irresolute function is contra-semi-$\alpha$-continuous.
Proof: Let $f : X \to Y$ be a contra-semi-$\alpha$-irresolute function. Let $V$ be an open set in $Y$. Then by Theorem 2.8(ii), $V$ is semi-$\alpha$-open in $Y$. Since $f$ is contra-semi-$\alpha$-irresolute, $f^{-1}(V)$ is semi-$\alpha$-closed in $X$. Hence $f$ is contra-semi-$\alpha$-continuous.

Theorem 4.11: For a function $f : X \to Y$, the following are equivalent:
(i) $f$ is contra-semi-$\alpha$-irresolute.
(ii) The inverse image of each semi-$\alpha$-closed set in $Y$ is semi-
$\alpha$-open in $X$.
(iii) For each $x \in X$ and each semi-$\alpha$-closed set $F$ in $Y$ with $f(x) \in F$, there exists a semi-$\alpha$-open set $U$ in $X$ such that $x \in U \subseteq F$.
(iv) $f(s*aCl(f^{-1}(F))) = f(s*aCl(f^{-1}(F)))$ for every semi-$\alpha$-
open set $F$ in $Y$.
(v) $f(s*aCl(f^{-1}(V))) = f(s*aCl(f^{-1}(V)))$ for every semi-$\alpha$-
open set $V$ in $Y$.
Proof: (i) $\Rightarrow$ (ii): Let $F$ be a semi-$\alpha$-closed set in $Y$. Then $Y \setminus F$ is semi-$\alpha$-open in $Y$. Since $f$ is contra-semi-$\alpha$-irresolute, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is semi-$\alpha$-closed in $X$.
(ii) $\Rightarrow$ (iii): Let $F$ be a semi-$\alpha$-closed set in $Y$ containing $f(x)$. Then $f^{-1}(F)$ is a semi-$\alpha$-open set containing $x$ such that $f(x) \not\subseteq F$.
(iii) $\Rightarrow$ (iv): Let $f$ be a semi-$\alpha$-closed set in $Y$ and $x \in f^{-1}(F)$, then $f(x) \in F$. By assumption, there is a semi-$\alpha$-open set $U$ in $X$ containing $x$ such that $x \in U \subseteq F$ which implies that $x \in U \subseteq f^{-1}(F)$. This follows that $f^{-1}(F) = U \cup \{x \in U : f^{-1}(F)\}$. By Theorem 2.10(i), $f^{-1}(F)$ is semi-$\alpha$-
open in $X$. By Theorem 2.11, $f(s*aCl(f^{-1}(F))) = f(s*aCl(f^{-1}(F)))$.
(iv) $\Rightarrow$ (v): Let $f$ be a semi-$\alpha$-closed set in $Y$. Then $Y \setminus f$ is semi-$\alpha$-closed in $Y$. By assumption, $f(s*aCl(f^{-1}(V))) = f(s*aCl(f^{-1}(V)))$. Taking the complements we get, $f(s*aCl(f^{-1}(V))) = f(s*aCl(f^{-1}(V)))$.
(v) $\Rightarrow$ (i): Let $V$ be any semi-$\alpha$-open set in $Y$. Then by assumption, $f(s*aCl(f^{-1}(V))) = f(s*aCl(f^{-1}(V)))$. By Theorem 2.12, $f^{-1}(V)$ is semi-$\alpha$-closed in $X$. 


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Theorem 4.12: (i) Every strongly semi*-u-irresolute function is semi*-u-irresolute and hence semi*-a-continuous.
(ii) Every semi a*-continuous function is strongly semi*-u-irresolute.

Proof: Let \( f: X \to Y \) be a semi*-a-open function. Then \( f^{-1}(V) \) is semi*-a-open for every semi*-a-open set \( V \subseteq Y \). Hence \( f^{-1}(V) \) is semi*-u-open. Therefore \( f \) is semi*-u-irresolute. Hence by Theorem 4.6, \( f \) is semi*-a-continuous.

Theorem 4.13: Every constant function is strongly semi*-a-irresolute.

Proof: Let \( f: X \to Y \) be a constant function defined by \( f(x) = x_0 \) for all \( x \in X \), where \( x_0 \) is a fixed point in \( Y \). Let \( V \subseteq Y \) be a semi*-a-open set in \( Y \). Then \( f^{-1}(V) = \emptyset \) or \( \emptyset \) according as \( x_0 \notin V \) or \( x_0 \in V \). Thus \( f^{-1}(V) \) is open in \( X \). Hence \( f \) is strongly semi*-a-irresolute.

Theorem 4.14: Let \( f: X \to Y \) be a function. Then the following are equivalent:
(i) \( f \) is strongly semi*-u-irresolute.
(ii) \( f^{-1}(F) \) is closed in \( X \) for every semi*-a-closed set \( F \) in \( Y \).
(iii) \( f(\text{Cl}(A)) \subseteq \text{s*aCl}(f(A)) \) for every subset \( A \) of \( X \).
(iv) \( f^{-1}(f(B)) \subseteq f^{-1}(s*aCl(B)) \) for every subset \( B \) of \( Y \).

Proof: (i) \( \Rightarrow \) (ii): Let \( f \) be a semi*-a-closed set in \( X \). Then \( f^{-1}(V) \) is semi*-a-open. Therefore \( f^{-1}(V) \) is open. Hence \( f \) is semi*-u-irresolute. 
(ii) \( \Rightarrow \) (i): Let \( A \subseteq X \). Let \( f \) be a semi*-u-irresolute set containing \( A \). Then by (ii), \( f^{-1}(F) \) is a closed set containing \( A \). This implies that \( Cl(A) \subseteq f^{-1}(F) \) and hence \( f(Cl(A)) \subseteq f(F) \). Therefore \( f(Cl(A)) \subseteq s*aCl(f(A)) \).

(iii) \( \Rightarrow \) (iv): Let \( B \subseteq Y \) and let \( A = f^{-1}(B) \). By assumption, \( (Cl(A)) \subseteq s*aCl(f(A)) \subseteq s*aCl(B) \). This implies that \( Cl(A) \subseteq f^{-1}(s*aCl(B)) \).
(iv) \( \Rightarrow \) (ii): Let \( f \) be semi*-u-closed in \( X \). Then by Theorem 2.13(ii), \( s*aCl(\text{Cl}(A))=f^{-1}(s*aCl(B)) \). This proves (iv).

(iii) \( \Rightarrow \) (v): Let \( f \) be a semi*-u-closed set in \( X \). Then \( \text{Cl}(f^{-1}(Y)) \subseteq Y \). By assumption, \( f^{-1}(Y) \subseteq f^{-1}(U) \) is semi*-a-closed in \( X \). Hence \( f^{-1}(U) \) is open. This proves (ii).

Theorem 5.1: A function \( f: X \to Y \) is said to be semi*-a-open if \( f(U) \) is semi*-a-open in \( Y \) for every open set \( U \) in \( X \).

Theorem 5.2: A function \( f: X \to Y \) is said to be contra-semi*-a-open if \( f(U) \) is semi*-a-closed in \( Y \) for every open set \( U \) in \( X \).

Theorem 5.3: A function \( f: X \to Y \) is said to be pre-semi*-a-open if \( f(U) \) is semi*-a-closed in \( Y \) for every semi*-a-open set \( U \) in \( X \).

Theorem 5.4: A function \( f: X \to Y \) is said to be contra-pre-semi*-a-open if \( f(U) \) is semi*-a-closed in \( Y \) for every semi*-a-open set \( U \) in \( X \).

Theorem 5.5: A function \( f: X \to Y \) is said to be semi*-a-closed if \( f(U) \) is semi*-a-closed in \( Y \) for every open set \( U \) in \( X \).

Theorem 5.6: A function \( f: X \to Y \) is said to be contra-semi*-a-closed if \( f(U) \) is semi*-a-closed in \( Y \) for every semi*-a-open set \( U \) in \( X \).

Theorem 5.7: A function \( f: X \to Y \) is said to be pre-semi*-a-closed if \( f(U) \) is semi*-a-closed in \( Y \) for every semi*-a-closed set \( U \) in \( X \).

Theorem 5.8: A function \( f: X \to Y \) is said to be contra-pre-semi*-a-closed if \( f(U) \) is semi*-a-closed in \( Y \) for every semi*-a-closed set \( U \) in \( X \).

Theorem 5.9: A bijection \( f: X \to Y \) is called a semi*-a-homeomorphism if \( f \) is both semi*-a-irresolute and pre-semi*-a-open. The set of all semi*-a-homeomorphisms of \( (X, \tau) \) into itself is denoted by \( \text{s*aH}(X, \tau) \).

Theorem 5.10: A function \( f: X \to Y \) is said to be semi*-a-totally continuous if \( f^{-1}(V) \) is clopen in \( X \) for every semi*-a-open set \( V \) in \( Y \).

Theorem 5.11: A function \( f: X \to Y \) is said to be totally semi*-a-continuous if \( f^{-1}(V) \) is semi*-a-regular in \( X \) for every open set \( V \) in \( Y \).

Theorem 5.12: (i) Every pre-semi*-a-open function is semi*-a-open.
(ii) Every semi*-a-open function is semi*-a-continuous.
(iii) Every contra-pre-semi*-a-open function is contra-semi*-a-continuous.
(iv) Every pre-semi*-a-closed function is semi*-a-closed.
(v) Every contra-pre-semi*-a-closed function is contra-semi*-a-closed.
Proof: Follows from definitions, Theorem 2.8 and Remark 2.9.

Theorem 5.13: Let \( f : X \rightarrow Y \) and be \( g : Y \rightarrow Z \) be functions. Then (i) \( g \circ f \) is pre-semi\( ^* \alpha \)-open if both \( f \) and \( g \) are pre-semi\( ^* \alpha \)-open.
(ii) \( g \circ f \) is semi\( ^* \alpha \)-open if \( f \) is semi\( ^* \alpha \)-open and \( g \) is pre-semi\( ^* \alpha \)-open.
(iii) \( g \circ f \) is pre-semi\( ^* \alpha \)-closed if both \( f \) and \( g \) are pre-semi\( ^* \alpha \)-closed.
(iv) \( g \circ f \) is semi\( ^* \alpha \)-closed if both \( f \) is semi\( ^* \alpha \)-closed and \( g \) is pre-semi\( ^* \alpha \)-closed.

Proof: Follows from definitions.

Theorem 5.14: Let \( f : X \rightarrow Y \) be a function where \( X \) is an Alexandroff space and \( Y \) is any topological space. Then the following are equivalent:
(i) \( f \) is semi\( ^* \alpha \)-totally continuous.
(ii) For each \( x \in X \) and each semi\( ^* \alpha \)-open set \( U \in Y \) with \( f(x) \in U \), there exists a clopen set \( V \) in \( X \) such that \( x \in V \) and \( f(V) \subseteq U \).

Proof: (i)\( \Rightarrow \)(ii): Suppose \( f : X \rightarrow Y \) is semi\( ^* \alpha \)-totally continuous. Let \( x \in X \) and let \( V \) be a semi\( ^* \alpha \)-open set containing \( f(x) \). Then \( V = f^{-1}(V) \) is a clopen set in \( X \) containing \( x \) and hence \( f(U) \subseteq V \).

(ii)\( \Rightarrow \)(i): Let \( V \) be a semi\( ^* \alpha \)-open set in \( Y \). Let \( x \in f^{-1}(V) \). Then \( f(x) \) lies in \( V \) and hence \( f^{-1}(V) \subseteq U \). Therefore \( f^{-1}(V) \) is open.

Theorem 5.15: A function \( f : X \rightarrow Y \) is semi\( ^* \alpha \)-totally continuous if and only if \( f^{-1}(F) \) is clopen in \( X \) for every semi\( ^* \alpha \)-closed set \( F \) in \( Y \).

Proof: Follows from definitions.

Theorem 5.16: A function \( f : X \rightarrow Y \) is totally semi\( ^* \alpha \)-continuous if and only if \( f \) is both semi\( ^* \alpha \)-continuous and contra-semi\( ^* \alpha \)-continuous.

Proof: Follows from definitions.

Theorem 5.17: A function \( f : X \rightarrow Y \) is semi\( ^* \alpha \)-totally continuous if and only if \( f \) is both strongly semi\( ^* \alpha \)-irresolute and contra-strongly semi\( ^* \alpha \)-irresolute.

Proof: Follows from definitions.

Theorem 5.18: Let \( f : X \rightarrow Y \) be semi\( ^* \alpha \)-totally continuous and \( A \) is a subset of \( Y \). Then the restriction \( f_A : A \rightarrow Y \) is semi\( ^* \alpha \)-totally continuous.

Proof: Let \( V \) be a semi\( ^* \alpha \)-open set in \( Y \). Then \( f^{-1}(V) \) is clopen in \( X \) and hence \( f(A \cap f^{-1}(V)) = A \cap f^{-1}(V) \) is clopen in \( A \). Hence the theorem follows.

Theorem 5.19: Let \( f : X \rightarrow Y \) be a bijection. Then the following are equivalent: (i) \( f \) is semi\( ^* \alpha \)-irresolute.
(ii) \( f^{-1} \) is pre-semi\( ^* \alpha \)-open.
(iii) \( f^{-1} \) is pre-semi\( ^* \alpha \)-closed.

Proof: Follows from definitions.

Theorem 5.20: A bijection \( f : X \rightarrow Y \) is a semi\( ^* \alpha \)-homeomorphism if and only if \( f \) and \( f^{-1} \) are semi\( ^* \alpha \)-irresolute.

Proof: Follows from definitions.

Theorem 5.21: (i) The composition of two semi\( ^* \alpha \)-homeomorphisms is a semi\( ^* \alpha \)-homeomorphism
(ii) The inverse of a semi\( ^* \alpha \)-homeomorphism is also a semi\( ^* \alpha \)-homeomorphism.

Proof: (i) Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be semi\( ^* \alpha \)-homeomorphisms. By Theorem 4.16 and theorem 5.13(i), \( g \circ f \) is a semi\( ^* \alpha \)-homeomorphism.
(ii) Let \( f : X \rightarrow Y \) be a semi\( ^* \alpha \)-homeomorphism. Then by Theorem 4.16(ii) and by Theorem 5.20, \( f^{-1} : Y \rightarrow X \) is also semi\( ^* \alpha \)-homeomorphism.

Theorem 5.22: If \( (X, \tau) \) is a topological space, then the set \( s^* \alpha H(X, \tau) \) of all semi\( ^* \alpha \)-homeomorphisms of \( (X, \tau) \) into itself forms a group.

Proof: Since the identity mapping \( I \) on \( X \) is a semi\( ^* \alpha \)-homeomorphism, \( I \in s^* \alpha H(X, \tau) \) and hence \( s^* \alpha H(X, \tau) \) is non-empty and the theorem follows from Theorem 5.21.

6. REFERENCES